APPENDIX E

This Appendix contains the CARA-based foundation of our model, and a complete proof of Proposition 4. Because this material is not as important as the other proofs, it is written as a separate supplement.

E.1 The CARA Setting

Agents can invest in a riskless asset with return r and in two risky assets paying the same cash flow. Cash flow is described by the cumulative dividend process

$$dD_t = \delta dt + \sigma dB_t,$$

where δ and σ are positive constants, and B_t is a standard Brownian motion. Agents derive utility from the consumption of a numéraire good, and have a CARA utility function

$$-E\left[\int_0^\infty \exp\left(-\alpha c_t - \beta t\right)\, dt\right]. \tag{E.1}$$

Each agent receives a cumulative endowment process

$$de_t = \sigma_e\left[\rho_t dB_t + \sqrt{1-\rho_t^2} dZ_t\right],$$

where σ_e is a positive constant, Z_t a standard Brownian motion independent of B_t, and ρ_t the instantaneous correlation between the dividend process and the endowment process. The process ρ_t can take three values: $\rho_t = -\overline{\rho} < 0$ for high-valuation agents, $\rho_t = \underline{\rho} > 0$ for low-valuation agents, and $\rho_t = 0$ for average-valuation agents. The processes (ρ_t, Z_t) are pairwise independent across agents. We set $A \equiv r\alpha$, $y \equiv A\sigma^2/2$, $\overline{x} \equiv A\overline{\rho}\sigma\sigma_e$, and $\underline{x} \equiv A\underline{\rho}\sigma\sigma_e$.

E.1.1 Walrasian Equilibrium

Under Assumptions 1 and 2, the Walrasian equilibrium is identical to that in Proposition 3. This is true even when agents are allowed to invest in integer multiples of one share and in both assets simultaneously, provided that we make the additional assumption

Assumption 3. $4y > \overline{x} + \underline{x}$.

Proposition 13. *Suppose that agents have CARA preferences and can hold any position* $(q_1, q_2) \in \mathbb{Z}^2$ *in the two assets. In a Walrasian equilibrium both assets trade at the same price*

$$p = \frac{\delta + \overline{x} - y}{r}$$

and the lending fee w *is zero. Moreover, high-valuation agents buy one share or stay out of the market, low-valuation agents short one share, and average-valuation agents stay out of the market.*

Proof: The lending fee is zero by the same argument as in Proposition 3. An agent maximizes (E.1) subject to the budget constraint

$$dW_t = \left[rW_t - c_t + \sum_{i=1}^{2} (\delta - rp_i) q_{it} \right] dt + \left[\sigma \sum_{i=1}^{2} q_{it} + \rho_t \sigma_e \right] dB_t + \sigma_e \sqrt{1 - \rho_t^2} dZ_t$$

and the transversality condition

$$\lim_{T \to \infty} E \left[\exp(-AW_T - \beta T) \right] = 0, \tag{E.2}$$

where W_t is the wealth and q_{it} is the number of shares invested in asset $i \in \{1, 2\}$. The agent's controls are the consumption $c \in \mathbb{R}$ and the investments $(q_1, q_2) \in \mathbb{Z}^2$. Obviously, if $p_1 \neq p_2$ the agent can achieve infinite utility by demanding an infinite amount of assets, contradicting equilibrium. Thus, in equilibrium p_1 and p_2 must be equal. Denoting their common value by p and the aggregate investment in the risky assets by $q \equiv q_1 + q_2$, we can write the budget constraint as

$$dW_t = \left[rW_t - c_t + (\delta - rp) q_t \right] dt + \left[\sigma q_t + \rho_t \sigma_e \right] dB_t + \sigma_e \sqrt{1 - \rho_t^2} dZ_t.$$

The agent's value function $J(W_t, \rho_t)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{(c,q) \in \mathbb{R} \times \mathbb{Z}} \left\{ -\exp(-\alpha c) + \mathcal{D}^{(c,q)} J(W, \rho) - \beta J(W, \rho) \right\}, \tag{E.3}$$

where

$$\begin{aligned} \mathcal{D}^{(c,q)} J(W, \rho) \quad &\equiv \quad J_W(W, \rho) \left[rW - c + (\delta - rp) q \right] + \frac{1}{2} J_{WW}(W, \rho) \left[\sigma^2 q^2 + 2\rho\sigma\sigma_e q + \sigma_e^2 \right] \\ &+ \kappa(\rho) \left[J(W, 0) - J(W, \rho) \right], \end{aligned}$$

and where the transition intensity $\kappa(\rho)$ is zero for $\rho = 0$, $\overline{\kappa}$ for $\rho = \overline{\rho}$, and $\underline{\kappa}$ for $\rho = \underline{\rho}$. We guess that $J(W, \rho)$ takes the form

$$J(W,\rho) = -\frac{1}{r}\exp\left[-A[W + V(\rho)] + \frac{r - \beta + \frac{A^2\sigma_e^2}{2}}{r}\right],$$

for some function $V(\rho)$. Replacing into (E.3), we find that the optimal consumption is

$$c(\rho) = r[W + V(\rho)] - \frac{r - \beta + \frac{A^2\sigma_e^2}{2}}{A}$$

and the optimal investment satisfies

$$q(\rho) \in \text{argmax}_{q\in\mathbb{Z}}\{C(\rho, q) - rpq\} \equiv Q(\rho),$$

where $C(\rho, q)$ is the incremental certainty equivalent of holding q shares relative to holding none. Using the definitions of y, $\overline{x}$, $\underline{x}$, we can write the certainty equivalent as $C(\overline{\rho}, z) = (\delta + \overline{x})q - yq^2$ for high-valuation agents, $C(\underline{\rho}, z) = (\delta - \underline{x})q - yq^2$ for low-valuation agents, and $C(0, z) \equiv \delta q - yq^2$ for average-valuation agents.

Plugging $c(\rho)$ back into (E.3), we find that (E.3) is satisfied iff

$$0 = -rV(\rho) + \max_{q\in\mathbb{Z}}\{C(\rho, q) - rpq\} + \kappa(\rho)\frac{1 - e^{-A(V(0)-V(\rho))}}{A}. \tag{E.4}$$

Equation (E.4) implies that $V(0) = \max_q\{C(0, q) - rpq\}/r$. Moreover, given $V(0)$, the equations for $V(\overline{\rho})$ and $V(\underline{\rho})$ are in only one unknown, and it is easy to check that they have a unique solution.

We next determine the equilibrium value of p. Because each type-ρ agent holds a position $q(\rho) \in Q(\rho)$, the average position $q_m(\rho)$ of these agents is in the convex hull of $Q(\rho)$. Market clearing requires that $q_m(0) = 0$ because average-valuation agents are in infinite measure. It also requires that

$$\frac{\overline{F}}{\overline{\kappa}}q_m(\overline{\rho}) + \frac{\underline{F}}{\underline{\kappa}}q_m(\underline{\rho}) = 2S. \tag{E.5}$$

Because the function $q \to C(\rho, q) - rpq$ is strictly concave, the set $Q(\rho)$ consists of either one or two elements. If there exists a q such that

$$C(\rho, q) - rpq > \max\left\{C(\rho, q + 1) - rp(q + 1), C(\rho, q - 1) - rp(q - 1)\right\}, \tag{E.6}$$

then this q is unique and $Q(\rho) = \{q\}$. Otherwise, there exists a unique q such that

$$C(\rho, q) - rpq = C(\rho, q + 1) - rp(q + 1), \tag{E.7}$$

and $Q(\rho) = \{q, q + 1\}$. Using Assumptions 1 and 3 and the first-order conditions (E.6) and (E.7), it is easy to check that for $p = (\delta + \overline{x} - y)/r$, we have $Q(\overline{\rho}) = \{0, 1\}$, $Q(\underline{\rho}) = \{-1\}$, and $Q(0) = \{0\}$. Equation (E.5) then follows from Assumption 2, implying that $p = (\delta + \overline{x} - y)/r$ is an equilibrium price. It is the unique equilibrium price because if $p > (\delta + \overline{x} - y)/r$, then no agent would choose $q > 0$, and if $p < (\delta + \overline{x} - y)/r$ then high-valuation agents would choose $q \geq 1$, while other agents would choose at least as much as for $p = (\delta + \overline{x} - y)/r$. ∎

E.1.2 Search Equilibrium

Proposition 14 studies agents' optimization problem in a general Poisson setting, and shows that the value function is of the form

$$J(W, \tau) = -\frac{1}{r} \exp\left[-A[W + V(\tau)] + \frac{r - \beta + \frac{A^2\sigma_e^2}{2}}{r}\right], \tag{E.8}$$

where $V(\tau)$ is a function characterized by (E.9). Using (E.9), it is easy to check that when α converges to zero, holding $(y, \overline{x}, \underline{x})$ constant, $V(\tau)$ satisfies the flow-value equations derived under the utility specification of Section 2. Therefore, if the trading strategies in the equilibria of Propositions 5 and 7 involve strict preferences (which is the case generically), they are also optimal under CARA preferences for small α. This means that the equilibria of Propositions 5 and 7 are also equilibria under CARA preferences.

Proposition 14. *Suppose that*

(i) An agent can be of finitely many types $\tau \in \mathcal{T}$.

(ii) While being of type τ, the agent receives a payoff described by the cumulative process

$$dX(\tau, t) = m(\tau)dt + \sqrt{\sigma(\tau_t)^2 + \sigma_e^2}\, d\tilde{B}_t,$$

where $\tilde{B}_t$ is a standard Brownian motion.

(iii) Transitions across types occur at the arrival times of a K-dimensional counting process N_t, with intensity associated to dimension k equal to a constant $\gamma(k)$. At the arrival times associated to dimension k, the agent can choose between types $\tau' \in \mathcal{T}'(\tau, k) \subseteq \mathcal{T}$.

(iv) Transition to type τ' brings an instant payoff $P(\tau, \tau')$.

Then, the value function is given by (E.8) with

$$rV(\tau) = m(\tau) - \frac{A}{2}\sigma(\tau)^2 + \sum_{k=1}^{K} \gamma(k) \max_{\tau' \in \mathcal{T}'(\tau,k)} \frac{1 - e^{-A\left[V(\tau') - V(\tau) + P(\tau,\tau')\right]}}{A}. \tag{E.9}$$

Proof: The agent's wealth evolves according to the SDE

$$dW_t = [rW_t + m(\tau_t) - c_t]dt + \sqrt{\sigma(\tau_t)^2 + \sigma_e^2}d\tilde{B}_t + \sum_{k=1}^{K} P(\tau_{t-}, \tau_t)\, dN_t(k).$$

The agent chooses a transition and consumption policy to maximize (E.1) subject to the transversality condition (E.2). We also impose the boundedness condition

$$E\left[\int_0^T \exp(-zW_t)\, dt\right] < \infty$$

for all $T \geq 0$ and $z \in \{r\alpha, 2r\alpha\}$, because it is needed for the verification argument. The HJB equation is

$$0 \quad = \sup_{c \in \mathbb{R}, \tau' \in \mathcal{T}'(\tau,k)} \left\{ -\exp\left[-\alpha\, c(\tau)\right] + \mathcal{D}^{(c,\tau')} J(W, \tau) - \beta J(W, \tau) \right\}, \tag{E.10}$$

where

$$\begin{aligned} \mathcal{D}^{(c,\tau')} J(W, \tau) \quad \equiv \quad & J_W(W, \tau)\left[rW - c(\tau) + m(\tau)\right] + \frac{1}{2}\left[\sigma(\tau)^2 + \sigma_e^2\right] J_{WW}(W, \tau) \\ & + \sum_{k=1}^{K} \gamma(k) \Big[J\left[W + P(\tau, \tau'), \tau'\right] - J(W, \tau) \Big]. \end{aligned}$$

Substituting (E.8) in (E.10) and maximizing with respect to consumption, we find that (E.8) is a solution iff $V(\tau)$ solves (E.9). ∎

E.2 Complete Proof of Proposition 4

When $p_i < V_{ni}$, (A.11) and (A.12) are replaced by

$$
\begin{aligned}
rV_{\overline{n}i} &= \delta + \overline{x} - y + w_i + \overline{\kappa}\left(V_{ni} - V_{\overline{n}i}\right) + \underline{\kappa}\left(V_{\overline{\ell}i} - V_{\overline{n}i}\right), && \text{(E.11)}\\
rV_{\underline{n}i} &= -\delta + \underline{x} - y - w_i + \underline{\kappa}\left(-p_i - V_{\underline{n}i}\right). && \text{(E.12)}
\end{aligned}
$$

The counterpart of (A.14) is

$$
(r+\underline{\kappa})\Sigma_i = \overline{x} + \underline{x} - 2y - (1-\theta)\sum_{j=1}^{2}\nu_j\mu_{\overline{\ell}j}\Sigma_j + \overline{\kappa}(V_{ni} - V_{\overline{n}i}). \tag{E.13}
$$

Subtracting (A.9) from (E.11), we find

$$
V_{\overline{n}i} - V_{ni} = \frac{\overline{x}}{r+\overline{\kappa}+\underline{\kappa}}, \tag{E.14}
$$

and can rewrite (E.13) as

$$
(r+\underline{\kappa})\Sigma_i = \overline{x} + \underline{x} - 2y - (1-\theta)\sum_{j=1}^{2}\nu_j\mu_{\overline{\ell}j}\Sigma_j - \frac{\overline{\kappa x}}{r+\overline{\kappa}+\underline{\kappa}}. \tag{E.15}
$$

Suppose that $\Sigma_1, \Sigma_2 \leq 0$. Then, a borrower and a lender of asset i are better off agreeing on a repo contract with a fee $w_i \approx 0$. Indeed, since $rp_i = \delta + \overline{x} - y$ from (A.8), we have $\delta - y + w_i - rp_i \approx -\overline{x} < 0$ and thus $V_{ni} < p_i$. Therefore, the surplus Σ_i under this contract is given by (A.14) and is positive. The lender is better off because of the fee, and if the fee is small the borrower is better off because $\Sigma_i > 0$. Therefore, $\Sigma_1, \Sigma_2 \leq 0$ cannot be part of an equilibrium.

Suppose that $\Sigma_1 > 0$ and $\Sigma_2 \leq 0$. Then, a borrower and a lender of asset 2 are better off agreeing on a repo contract with a fee $w_2 \approx 0$. Indeed, the surplus Σ_2 under this contract is given by (A.14). If Σ_1 is given by (A.14), then $\Sigma_2 = \Sigma_1 > 0$. If Σ_1 is given by (E.13), then $\Sigma_2 = \overline{x}/(r+\overline{\kappa}+\underline{\kappa}) > 0$. Therefore, $\Sigma_1 > 0$ and $\Sigma_2 \leq 0$ cannot be part of an equilibrium, and the only possible outcome is $\Sigma_1, \Sigma_2 > 0$ and $\nu_1 = \nu_2 = \nu$.

Since $\nu_1 = \nu_2 = \nu$, the Law of One Price holds if $p_i \geq V_{ni}$ for both assets or $p_i < V_{ni}$ for both assets. Consider an equilibrium in which $p_1 \geq V_{n1}$ and $p_2 < V_{n2}$. Then (A.8) and (A.13) imply that

$$rp_i = \delta + \overline{x} - y + \nu\mu_{\underline{bo}}\theta\Sigma_i, \tag{E.16}$$

(A.8), (A.11) and (A.13) imply that

$$w_1 = (r + \overline{\kappa} + \underline{\kappa} + \nu\mu_{\underline{bo}})\theta\Sigma_1, \tag{E.17}$$

(A.8), (A.13), (E.11) and (E.14) imply that

$$w_2 = (r + \underline{\kappa} + \nu\mu_{\underline{bo}})\theta\Sigma_2 + \frac{\overline{\kappa x}}{r + \overline{\kappa} + \underline{\kappa}}, \tag{E.18}$$

(E.16), (E.17) and $\delta - y + w_1 - rp_1 \leq 0$ imply that

$$(r + \overline{\kappa} + \underline{\kappa})\theta\Sigma_1 - \overline{x} \leq 0, \tag{E.19}$$

and (E.16), (E.18) and $\delta - y + w_2 - rp_2 > 0$ imply that

$$(r + \overline{\kappa} + \underline{\kappa})\theta\Sigma_2 - \overline{x} > 0. \tag{E.20}$$

Equations (E.19) and (E.20) imply that $\Sigma_2 > \Sigma_1$. But then, a borrower and a lender of asset 1 can be made better off agreeing to a contract with a fee $\tilde{w}_1 > w_1$ such that $\delta - y + \tilde{w}_1 - rp_1$ is slightly positive. Using (A.9), this implies that $\tilde{V}_{n1} > p_1$, so that the lender finds it optimal not to terminate when he reverts to an average valuation. Hence, this contract generates surplus Σ_2. Because $\delta - y + \tilde{w}_1 - rp_1$ is slightly positive, we also have that $\tilde{V}_{n1} \approx p_1$, meaning that a lender is nearly indifferent between terminating or not. This means that the change in the lender's utility is

$$\Delta V_{\overline{n}i} \approx \frac{\tilde{w}_1 - w_1}{r + \overline{\kappa} + \underline{\kappa}} > 0,$$

the PV of the lending fee difference assuming that the lender follows the same termination strategy than with w_1. The change in the borrower's utility is $\Sigma_2 - \Sigma_1 - \Delta V_{\overline{n}i}$. Factoring out $1/(r + \overline{\kappa} + \underline{\kappa})$, we can write this as

$$\begin{aligned}
& (r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) - (\tilde{w}_1 - w_1) \\
\approx \; & (r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) - \left[rp_1 - \delta + y - (r + \overline{\kappa} + \underline{\kappa} + \nu\mu_{\underline{bo}})\theta\Sigma_1\right] \\
= \; & (r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) - \left[\delta + \overline{x} - y + \nu\mu_{\underline{bo}}\theta\Sigma_1 - \delta + y - (r + \overline{\kappa} + \underline{\kappa} + \nu\mu_{\underline{bo}})\theta\Sigma_1\right] \\
= \; & (r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) - [\overline{x} - (r + \overline{\kappa} + \underline{\kappa})\theta\Sigma_1] \\
= \; & (1 - \theta)(r + \overline{\kappa} + \underline{\kappa})(\Sigma_2 - \Sigma_1) + [(r + \overline{\kappa} + \underline{\kappa})\theta\Sigma_2 - \overline{x}] > 0.
\end{aligned}$$

Therefore, the conjectured equilibrium is not possible. ∎