Online Appendix for "Optimal Taxation and Human Capital Policies over the Life Cycle"

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A Implementation Proofs

A sequential reformulation of the recursive allocations:

Any solution to the recursive social planner's problem can be mapped into a solution which depends on the full past history, using a recursive construction. To see this, denote the solutions to the recursive problem at time t, for each realized type θ_t, as a function of all state variables by:

$$v_t^*\left(v_{t-1},\Delta_{t-1},s_{t-1},\theta_{t-1},\theta_t\right),\Delta_t^*\left(v_{t-1},\Delta_{t-1},s_{t-1},\theta_{t-1},\theta_t\right),\omega_t^*\left(v_{t-1},\Delta_{t-1},s_{t-1},\theta_{t-1},\theta_t\right),$$
$$y_t^*\left(v_{t-1},\Delta_{t-1},s_{t-1},\theta_{t-1},\theta_t\right),s_t^*\left(v_{t-1},\Delta_{t-1},s_{t-1},\theta_{t-1},\theta_t\right),c_t^*\left(v_{t-1},\Delta_{t-1},s_{t-1},\theta_{t-1},\theta_t\right)$$

and the solutions to the planner's sequential problem by $\left\{x_t^*\left(\theta^t\right)\right\}=\left\{y_t^*\left(\theta^t\right),s_t^*\left(\theta^t\right),c_t^*\left(\theta^t\right)\right\}$ The dependence on initial promised utilities in period 1, due to initial heterogeneity, is dropped for notational convenience; they can be just carried as an additional conditioning variable. This allocation gives rise to a sequence of utilities for the agent, generated recursively by:

$$\begin{aligned}
\omega_t^*\left(\theta^t\right) &= u_t\left(c_t^*\left(\theta^t\right)\right)-\phi_t\left(\frac{y_t^*\left(\theta^t\right)}{w_t\left(\theta_t,s_t^*\left(\theta^t\right)\right)}\right)+\beta\int\omega_{t+1}^*\left(\theta^t,\theta_{t+1}\right)f^{t+1}\left(\theta_{t+1}|\theta_t\right)d\theta_{t+1}\\
v_t^*\left(\theta^t\right) &= \int\omega_{t+1}^*\left(\theta^t,\theta_{t+1}\right)f^{t+1}\left(\theta_{t+1}|\theta_t\right)d\theta_{t+1}\\
\Delta_t^*\left(\theta^t\right) &= \int\omega_{t+1}^*\left(\theta^t,\theta_{t+1}\right)\frac{\partial}{\partial\theta_t}f^{t+1}\left(\theta_{t+1}|\theta_t\right)d\theta_{t+1}
\end{aligned}$$

To initialize the allocations, set $\omega_1^*\left(\theta_1\right)=\omega_1^*\left(v_0,\Delta_0,s_0,\theta_0,\theta_1\right)=\underline{U}\left(\theta_1\right)$ (if there is initial heterogeneity in θ_1, with v_0 arbitrary in that case, $\Delta_0=0$), $y_1^*\left(\theta_1\right)=y_1^*\left(v_0,\Delta_0,s_0,\theta_0,\theta_1\right)$, $s_1^*\left(\theta_1\right)=s_1^*\left(v_0,\Delta_0,s_0,\theta_0,\theta_1\right)$, and construct iteratively the full, history dependent allocation for all histories θ^t using the difference equations:

$$\omega_t^*\left(\theta^t\right)=\omega_t^*\left(v_{t-1}^*\left(\theta^{t-1}\right),\Delta_{t-1}^*\left(\theta^{t-1}\right),s_{t-1}^*\left(\theta^{t-1}\right),\theta_{t-1},\theta_t\right)$$

where v_{t-1}^* and Δ_{t-1}^* are written as functions of ω_{t-1}, i.e.,

$$v_{t-1}^*\left(\theta^{t-1}\right)=\frac{1}{\beta}\left[\omega_{t-1}^*\left(\theta^{t-1}\right)-u_{t-1}\left(c_{t-1}^*\left(\theta^{t-1}\right)\right)+\phi_{t-1}\left(\frac{y_{t-1}^*\left(\theta^{t-1}\right)}{w_{t-1}\left(\theta_{t-1},s_{t-1}^*\left(\theta^{t-1}\right)\right)}\right)\right]$$

$$\Delta_{t-1}^{*}\left(\theta^{t-1}\right)=\frac{1}{\beta}\left[\frac{\partial\omega_{t-1}^{*}\left(\theta^{t-1}\right)}{\partial\theta_{t-1}}-\phi_{t-1}^{\prime}(y_{t-1}^{*}\left(\theta^{t-1}\right)/w_{t-1}\left(s_{t-1}^{*}\left(\theta^{t-1}\right),\theta_{t-1}\right))\frac{y_{t-1}^{*}\left(\theta^{t-1}\right)}{w_{t-1}\left(s_{t-1}^{*}\left(\theta^{t-1}\right),\theta_{t-1}\right)^{2}}\frac{\partial w_{t-1}}{\partial\theta_{t-1}}\right]$$

Construct also:

$$\begin{aligned}
y_t^*\left(\theta^t\right) &= y_t^*\left(v_{t-1}^*\left(\theta^{t-1}\right),\Delta_{t-1}^*\left(\theta^{t-1}\right),s_{t-1}^*\left(\theta^{t-1}\right),\theta_{t-1}\right)\\
s_t^*\left(\theta^t\right) &= s_t^*\left(v_{t-1}^*\left(\theta^{t-1}\right),\Delta_{t-1}^*\left(\theta^{t-1}\right),s_{t-1}^*\left(\theta^{t-1}\right),\theta_{t-1}\right)\\
c_t^*\left(\theta^t\right) &= c_t^*\left(v_{t-1}^*\left(\theta^{t-1}\right),\Delta_{t-1}^*\left(\theta^{t-1}\right),s_{t-1}^*\left(\theta^{t-1}\right),\theta_{t-1}\right)
\end{aligned}$$

Using these sequential optimal policies in the planner's problem we can rewrite the costs as a function of the history, i.e., $K_t\left(\theta^{t-1}\right)=K_t\left(v_{t-1}^*\left(\theta^{t-1}\right),s_{t-1}^*\left(\theta^{t-1}\right),\theta_{t-1}\right)$. or, if agents also have heterogeneous initial wealth levels, $K_t\left(b_0,\theta^{t-1}\right)$. Note that in the planner's problem (i.e., in the direct revelation mechanism) initial wealth holdings are irrelevant since the planner can fully observe and allocate consumption. Since agents can borrow at the same rate as the government, without loss of generality, agents can do all the borrowing and saving on their own. The implicit wealth levels are then defined recursively as: $\frac{1}{R}b_t^*\left(b_0,\theta^t\right)=y_t^*\left(b_0,\theta^t\right)-c_t^*\left(b_0,\theta^t\right)-M\left(e_t^*\left(b_0,\theta^t\right)\right)+b_{t-1}^*\left(b_0,\theta^t\right)$.

Proof of Proposition (4)

In the first step, we construct the history-independent savings tax, in the spirit of Werning (2011), with added human capital. Consider an incentive compatible allocation expressed as a function of the full history $c_t\left(\theta^t\right)$, $y_t\left(\theta^t\right)$, $e_t\left(\theta^t\right)$,$s_t\left(\theta^t\right)$, and its associated continuation utility $\omega_t\left(\theta^t\right)$, and suppose it is implemented as the outcome of a direct revelation mechanism with no savings. Allow agents to save any desired amount, with the restriction $b_T\geq 0$ (end of period T asset level). Consider a general tax function $T_{Kt}\left(b_t,r^t\right)$ as a function of savings and the history of reports up to period t. Given the report of the agent up to period $t-1$, and the report of the current shock, the planner assigns $c_t\left(r^{t-1},r_t\right)$, $y_t\left(r^{t-1},r_t\right)$, $e_t\left(r^{t-1},r_t\right)$ and the agent chooses savings b_t. Let $V_t\left(b_{t-1},r^{t-1},\theta_t\right)$ be the continuation value of an agent with beginning of period savings b_{t-1}, a history of reports r^{t-1}, and a realized shock θ_t. The agent's problem is:

$$V_t\left(b_{t-1},r^{t-1},\theta_t\right)=\max_{r_t,b_t}\tilde{V}_t\left(b_{t-1},b_t,r^{t-1},r_t,\theta_t\right)$$

with $\tilde{V}_t\left(b_{t-1},b_t,r^{t-1},r_t,\theta_t\right)$ defined as the value from saving an amount b_t and reporting r_t:

$$\tilde{V}_t\left(b_{t-1},b_t,r^{t-1},r_t,\theta_t\right)=\left\{\begin{array}{c} u_t\left(c_t\left(r^{t-1},r_t\right)+b_{t-1}-\left(\frac{1}{R}b_t+T_{Kt}\left(b_t,r^{t-1},r_t\right)\right)\right)-\phi_t\left(\frac{y\left(r^{t-1},r_t\right)}{w_t\left(\theta_t,s_t\left(r^{t-1},r_t\right)\right)}\right)\\ +\beta E\left(V_{t+1}\left(b_t,r^t,\theta_{t+1}\right)|\theta_t\right)\end{array}\right\}$$

In period $T-1$, define for each type realization θ_{T-1}, current asset level b_{T-2}, history of reports r^{T-1}, and savings levels b_{T-1} a fictitious tax T_{Kt}^{θ} which makes an agent just indifferent between saving b_{T-1} and saving zero.

$$\begin{aligned}
\omega_{T-1}\left(\theta^{T-1}\right) &= u_{T-1}\left(c_{T-1}\left(r^{T-1}\right)+b_{T-2}-\frac{1}{R}b_{T-1}-T_{KT-1}^{\theta}\left(b_{T-1},r^{T-1},\theta_{T-1}\right)\right)\\
&\quad -\phi_{T-1}\left(\frac{y_{T-1}\left(r^{T-1}\right)}{w_{T-1}\left(\theta_{T-1},s_{T-1}\left(r^{T-1}\right)\right)}\right)+\beta E\left(V_T\left(b_{T-1},r^{T-1},\theta_T\right)|\theta_{T-1}\right)
\end{aligned}$$

Taking the sup over all types θ_{T-1} yields a history-dependent, but type-independent savings tax $T_K^r\left(b_{T-1}, r^{T-1}\right) = \sup_{\theta_{T-1}} T_{KT-1}^{\theta}\left(b_{T-1}, r^{T-1}, \theta_{T-1}\right)$.

By induction, suppose that in period t the agent is faced with a continuation value function $V_{t+1}\left(b_t, r^t, \theta_{t+1}\right)$. Define the tax function for period t as $T_{Kt}\left(b_t, r^t\right) = \sup_{\theta_t} T_{Kt}^{\theta}\left(b_t, r^t, \theta_t\right)$ with $T_{Kt}^{\theta}\left(b_t, r^t, \theta_t\right)$ to equate:

$$\omega_t\left(\theta^t\right) = u_t\left(c_t\left(r^t\right) + b_{t-1} - \frac{1}{R}b_t - T_{Kt}^{\theta}\left(b_t, r^t, \theta_t\right)\right) - \phi_t\left(\frac{y_t\left(r^t\right)}{w_t\left(\theta_t, s_t\left(r^t\right)\right)}\right) + \beta E\left(V_{t+1}\left(b_t, r^t, \theta_{t+1}\right)|\theta_t\right)$$

Work backwards to define the tax functions in this fashion for all periods. The sequence of tax functions $\left\{T_{Kt}\left(b_t, r^t\right)\right\}_{t=1}^{T-1}$ thus defined implement zero savings each period for all sequences of reports. Next, take the supremum over all histories of reports r^t to obtain a history independent tax which implements zero savings.

$$T_K\left(b\right) = \sup_{r^t} T_{Kt}\left(b, r^t\right)$$

In the second step, we construct the loan and repayment schedule that mimics the direct mechanism above. Note that L^t can directly be mapped into a history of human capital and education levels e^t (recall that $s_0 = 0$), using that $e_t = M_t^{-1}\left(L_t\right)$ Hence, $\left(L^{t-1}, y^{t-1}\right)$ and $\left(e^{t-1}, y^{t-1}\right)$ will be used interchangeably. First, set implicit finite (but potentially very large) upper and lower limits on asset holdings, $\bar{b} > 0$ and $\underline{b} < 0$. This can be done either by extending the proposed savings tax so that for $b_t > \bar{b}$ and $b_t < \underline{b}$, $\forall t$, it is confiscatory (e.g., tax away all wealth and imposes a large penalty on borrowing) or by directly setting borrowing and saving limits. Let $B \equiv \left[\underline{b}, \bar{b}\right]$.

In each period, all allocations which can arise as outcomes in the optimum are made affordable:

$$D_t\left(L^{t-1}, y^{t-1}, e_t^*\left(\theta^{t-1}, \theta\right), y_t^*\left(\theta^{t-1}, \theta\right)\right) + T_Y\left(y_t^*\left(\theta^{t-1}, \theta\right)\right) = y_t^*\left(\theta^{t-1}, \theta\right) - c_t^*\left(\theta^{t-1}, \theta\right) \quad (1)$$

$$L_t\left(e_t^*\left(\theta^{t-1}, \theta\right)\right) = M_t\left(e_t^*\left(\theta^{t-1}, \theta\right)\right) \quad (2)$$

for all $\left(L^{t-1}, y^{t-1}\right)$ such that $\theta^{t-1} \in \Theta^{t-1}\left(\left\{M_1^{-1}\left(L_1\right), ..., M_{t-1}^{-1}\left(L_{t-1}\right)\right\}, y^{t-1}\right) \neq \emptyset$ and all $\theta \in \Theta$.

To mimic the direct revelation mechanism, we need to exclude pairs $\left(e_t, y_t\right)$ which would not be assigned to any type θ_t in the social planner's problem after a history θ^{t-1}, and, consequently, exclude histories $\left(y^{t-1}, e^{t-1}\right)$, which do not correspond to any history θ^{t-1}. Call "non-allowed" a choice which is not assigned in the social planner's problem for *any* type θ_t after history θ^{t-1}, i.e., such that $\left(e_t, y_t\right) \notin Q_{e,y}^{t-1}\left(\theta^{t-1}\right)$. The repayments at non-allowed levels have to be sufficiently dissuasive to make them strictly dominated by allowed choices. Then the history of reports would exactly be tracked by $r^{t-1} = \theta^{t-1} \in Q_{e,y}^{t-1}\left(y^{t-1}, e^{t-1}\right)$, and the agent's problem becomes equivalent to making a report $r_t \in \Theta$ in each period, by choosing a pair $\left(e_t, y_t\right)$ designed for some θ_t after θ^{t-1}. We know that in this case, the previously constructed history-independent savings tax would enforce zero savings.

There are several ways to rule out non-allowed allocations, and the goal here is just to provide a possible one, which is to simply set the repayment prohibitively high, so that irrespective of savings, it is never optimal to chose such allocations. For instance, after a history $\theta^{t-1} \in \Theta^{T-1}\left(e^{T-1}, y^{T-1}\right)$ and for any choice $\left(e_t, y_t\right) \notin Q_{ey}^t\left(\theta^{t-1}\right)$, set

$$D_t\left(L^{t-1}, y^{t-1}, e_t, y_t\right) + T_Y\left(y_t\right) > \left[\bar{b} - \underline{b} + y_t\right]$$

i.e., the repayment plus income tax at least confiscate income and impose an additional penalty such that all wealth is confiscated and agents can never borrow sufficiently to retain positive consumption.[1] This leaves the agent with zero consumption, and will never be chosen. More generally, arbitrarily large repayments can be set. The second and less draconian way is to take the envelope of the repayment schedules which, after each history, for any possible beginning of period wealth and optimal savings choice, would make the agent just indifferent between any non-allowed allocation and his optimal allocation.[2] Whatever the method chosen, once the non-allowed choices are ruled out, each period, after every history, the agent faces a problem equivalent in outcomes to the direct revelation mechanism, i.e., he faces only allocations which are also available to him in the social planner's problem after that history. Accordingly, the savings tax ensures that he will find it optimal not to save. By temporal incentive compatibility, he will chose the allocation designed for him.

Proof of Proposition (5) :

The proof is an extended and modified version of the proof of Proposition in Albanesi and Sleet (2006), adding human capital. With iid shocks, the recursive formulation of the relaxed program no longer requires the states Δ and θ_{t-1}:

$$K(v, s_-, t) = \min_{(c(\theta),l(\theta),\omega(\theta),s(\theta),v(\theta))} \int \left[c(\theta) + M_t(s(\theta) - s_-) - w(\theta, s(\theta))\, l(\theta) + \frac{1}{R} K(v(\theta), s(\theta), t+1)\right] f(\theta)\, d\theta$$

subject to:

$$\begin{aligned} \omega(\theta) &= u_t(c(\theta)) - \phi_t(l(\theta)) + \beta v(\theta) \\ \dot{\omega}(\theta) &= \frac{w_{\theta,t}}{w_t} l(\theta)\, \phi_{l,t}(l(\theta)) \\ v &= \int \omega(\theta)\, f^t(\theta|\theta_-)\, d\theta \end{aligned}$$

Denote by $K^{-1,s}$ and $K^{-1,v}$ the partial inverse functions of $K(v, s, t)$ with respect to its arguments s and v respectively. Define the set

$$Q^t_{e,y}(b_-, s_-) = \left\{e, y : e = e^*_t(v, s_-, \theta_t), y = y^*_t(v, s_-, \theta_t) \text{ for some } \theta_t \in \Theta, \text{ with } v = K^{-1,v}(b_-, s_-)\right\}$$

to be the set of output levels y_t and education levels e_t which are available to an agent with promised utility v and previous human capital level s_- in the social planner's problem. The value function of the agent who starts a period with wealth b_{t-1} and human capital s_{t-1} is denoted by $V_t(b_{t-1}, s_{t-1})$, as in the main text.

In each period t, the agent's problem can be split into two stages, because of the separability between consumption and labor. In stage 1, he chooses labor supply l_t (equivalently, output y_t) and

[1] To extend the repayment scheme's domain to histories L^{t-1}, y^{t-1} for which $\Theta^{T-1}\left(e^{T-1}, y^{T-1}\right) = \emptyset$, set for all e_t, y_t after such histories:

$$D_t\left(L^{t-1}, y^{t-1}, e_t, y_t\right) + T_Y(y_t) > \bar{b} - \underline{b} + y_t$$

[2] A more sophisticated implementation, which smooths the repayment schedule to make it differentiable is currently explored by the author. That implementation involves adding wealth as a conditioning variable in the repayment function. See as well the next implementation below.

human capital expenses e_t. He pays a tax $T_t(b_{t-1}, s_{t-1}, y_t, e_t)$ and is left with a total resource amount $b_t^m = y_t - T_t(b_{t-1}, s_{t-1}, y_t, e_t) - M(e_t) + b_{t-1}$. In stage 2, he chooses consumption and next-period bond holdings, b_t to maximize $V_t^m(b_t^m, s_{t-1} + e_t)$, the intermediate value function from resource level b_t^m, defined as:

$$\begin{aligned} V_t^m(b_t^m, s_t) &= \max_{c_t, b_t}(u_t(c_t) + \beta V_{t+1}(b_t, s_t)) \\ s.t. &: \quad b_t^m = c_t + \frac{1}{R} b_t \end{aligned}$$

with $V_T^m(b_T^m, s_T) = u_T(b_T^m)$. Denote the market outcomes by $\hat{b}_t(b_t^m, s_t)$ and $\hat{c}_t(b_t^m, s_t)$. In stage 1, the problem of the agent is:

$$\begin{aligned} V_t(b_{t-1}, s_{t-1}) &= \max_{\{y_t(\theta), e_t(\theta), b_t^m(\theta)\}} \int_\Theta (-\phi_t(y_t(\theta)/w_t(\theta, s_{t-1} + e_t(\theta))) + V_t(b_t^m(\theta), s_t(\theta))) f(\theta) d\theta \\ s.t. &: \quad b_t^m = y_t - T_t(b_{t-1}, s_{t-1}, y_t, e_t) - M(e_t) + b_{t-1} \end{aligned}$$

Let the market outcomes be denoted by $\hat{y}_t(b_{t-1}, s_{t-1}, \theta_t)$, $\hat{e}_t(b_{t-1}, s_{t-1}, \theta_t)$, and $\hat{b}_t^m(b_{t-1}, s_{t-1}, \theta_t)$ for each type θ_t.

In each period, the planner solves a two-stage problem as well. In the first stage, he allocates human capital expenses e_t, output y_t, and an intermediate promised utility v_t^m. In the second stage, he allocates consumption c_t, and continuation utility v_t. In stage 2, given an intermediate promised utility v_t^m, and an acquired human capital level $s_{t-1} + e_t = s_t$, the planner solves the program with intermediate continuation cost function $K^m(v_t^m, s_t, t)$:

$$\begin{aligned} K^m(v_t^m, s_t, t) &= \min_{c_t, v_t}\left(c_t + \frac{1}{R} K(v_t, s_t, t+1)\right) \\ s.t. &: \quad u_t(c_t) + \beta v_t = v_t^m \end{aligned}$$

with $K(v_T, s_T, T+1) \equiv 0$. Denote the solutions to this problem by $c_t^*(v_t^m, s_t)$ and $v_t(v_t^m, s_t)$.

In stage 1, the problem of the planner is hence:

$$\begin{aligned} K(v_{t-1}, s_{t-1}, t) &= \min_{\{v_t^m(\theta), e_t(\theta), y_t(\theta)\}} \int_\Theta (M(e_t(\theta)) - y_t(\theta) + K^m(v_t^m(\theta), s_{t-1} + e_t(\theta), t)) f(\theta) d\theta \\ s.t. &: \quad v_t^m(\theta) - \phi_t(y_t(\theta)/w_t(\theta, s_{t-1} + e_t(\theta))) \\ &\geq \quad v_t^m(\theta') - \phi_t\left(y_t(\theta')/w_t\left(\theta, s_{t-1} + e_t(\theta')\right)\right) \quad \forall \theta, \theta' \\ & \quad \int_\Theta (v_t^m(\theta) - \phi_t(y_t(\theta)/w_t(\theta, s_{t-1} + e_t(\theta)))) f(\theta) d\theta = v_{t-1} \end{aligned}$$

Denote the solutions to the planner problem by $v_t^{m*}(v_{t-1}, s_{t-1}, \theta_t)$, $e_t^*(v_{t-1}, s_{t-1}, \theta_t)$, and $y_t^*(v_{t-1}, s_{t-1}, \theta_t)$.

In stage 2, the problem of an agent who has acquired human capital s_t and who starts with intermediate wealth $b_t^m = K^m(v_t^m, s_t, t)$ is exactly the dual of the planner's problem who has promised utility $v_t^m = (K^m)^{-1,v}(b_t^m, s_t, t)$ for an agent with human capital s_t (where $(K^m)^{-1,v}$ is the partial inverse of K^m with respect to its first argument), so that the consumption choice of the agent and planner will coincide if mapped appropriately, i.e., $\hat{c}_t(K^m(v_t^m, s_t, t), s_t) = c_t^*(v_t^m, s_t)$. Furthermore, $\hat{b}_t(K^m(v_t^m, s_t, t), s_t) = K(v_t(v_t^m, s_t), s_t, t+1)$. To see this, suppose instead that there was another pair $\left(\tilde{c}, \tilde{b}\right) \neq (c^*, K)$ such that $\tilde{c} + \frac{1}{R}\tilde{b} = b_t^m$, but which yields higher utility: $u_t(\tilde{c}) + \beta V_{t+1}\left(\tilde{b}, s_t\right) > v_t^{m*} = (K^m)^{-1,v}(b_t^m, s_t, t)$ Note that the choice of s_t, already made in the previous stage is now fixed. Under the assumption that $V_{t+1}(., s_t)$ is increasing and continuous in its first argument, and given that $u_t(c)$ is increasing and continuous in c, there is also a pair $\left(\tilde{c}', \tilde{b}'\right)$ such that $\tilde{c} \leq c^*$, $\tilde{b} \leq K$ with one or both of these inequalities strict and such that $u_t(\tilde{c}) + \beta V_{t+1}\left(\tilde{b}, s_t\right) = v_t^{m*}$. But then $\left(\tilde{c}', \tilde{b}'\right)$ is better than (c^*, K) in the planner's problem and hence, (c^*, K) could not have been optimal, a contradiction.

For the first stage, consider an agent with initial wealth and human capital levels b_{t-1} and s_{t-1}. First, map the allocations from the social planner's problem to allocations defined on the state space $(b_{t-1}, s_{t-1}, \theta_t)$:

$$\begin{aligned} y_t^*(b_{t-1}, s_{t-1}, \theta_t) &= y_t^*\left(K^{-1,v}(b_{t-1}, s_{t-1}, t), s_{t-1}, \theta_t\right) \\ e_t^*(b_{t-1}, s_{t-1}, \theta_t) &= e_t^*\left(K^{-1,v}(b_{t-1}, s_{t-1}, t), s_{t-1}, \theta_t\right) \\ b_t^{*m}(b_{t-1}, s_{t-1}, \theta_t) &= K_t^m\left(v_t^{m*}(v_{t-1}, s_{t-1}, \theta_t), s_{t-1} + e_t^*(v_{t-1}, s_{t-1}, \theta_t), t\right) \end{aligned}$$

Then, set the tax level such that for all $y, e \in Q_{e,y}^t(b_{t-1}, s_{t-1})$,

$$T_t(b_{t-1}, s_{t-1}, y_t^*(b_{t-1}, s_{t-1}, \theta_t), e_t^*(b_{t-1}, s_{t-1}, \theta_t)) = y_t^*(b_{t-1}, s_{t-1}, \theta_t) - b_t^{*m}(b_{t-1}, s_{t-1}, \theta_t) + b_{t-1} - M_t(e_t^*(b_{t-1}, s_{t-1}, \theta_t))$$

(that is, make all allocations consistent with an allocation in the planner's problem just affordable).

To extend the definition of the tax function to the full domain of allocations, even those which would not arise in the social planner's problem, three steps are taken. First, to rule out wealth levels b_{t-1} not observed along the equilibrium path, i.e., such that $b_{t-1} \neq K\left(v_{t-1}^*\left(\theta^{t-1}\right), s_{t-1}^*\left(\theta^{t-1}\right), t\right)$ for any θ^{t-1}, set the borrowing limits to be $\underline{b_{t-1}} = \min_{v,s} K(v, s, t)$ where the min is taken over the possible values of v and s at time t in the planner's program. Second, if $b_{t-1} = K\left(v_{t-1}^*\left(\tilde{\theta}^{t-1}\right), s_{t-1}^*\left(\tilde{\theta}^{t-1}\right), t\right)$ for some $\tilde{\theta}^{t-1}$ but $s_{t-1} \neq s_{t-1}^*\left(\tilde{\theta}^{t-1}\right)$ (that is, the levels of wealth and human capital from the past period are not mutually consistent), set the tax T_t such that, for all e_t, y_t:

$$T_t(b_{t-1}, s_{t-1}, e_t, y_t) = y_t + \max\{b_{t-1}, 0\} - \min\{\underline{b_t}, 0\}$$

Finally, if the agent is on the equilibrium path with b_{t-1} and s_{t-1}, and letting $v_{t-1} = K^{-1,v}(b_{t-1}, s_{t-1}, t)$, set T_t such that for all pairs $y_t, e_t \notin Q_{e,y}^t(b_{t-1}, s_{t-1})$, and all θ_t

$$\begin{aligned} & -\phi_t\left(\frac{y_t}{w_t(\theta_t, s_{t-1} + e_t)}\right) + \beta V_t^m(b_{t-1} + y_t - T_t(b_{t-1}, s_{t-1}, y_t, e_t) - M_t(e_t), s_{t-1} + e_t) \\ < \; & -\phi_t\left(\frac{y_t^*(v_{t-1}, s_{t-1}, \theta_t)}{w_t(\theta_t, s_{t-1} + e_t^*(v_{t-1}, s_{t-1}, \theta_t))}\right) + v_t^{m*}(v_{t-1}, s_{t-1}, \theta_t) \end{aligned}$$

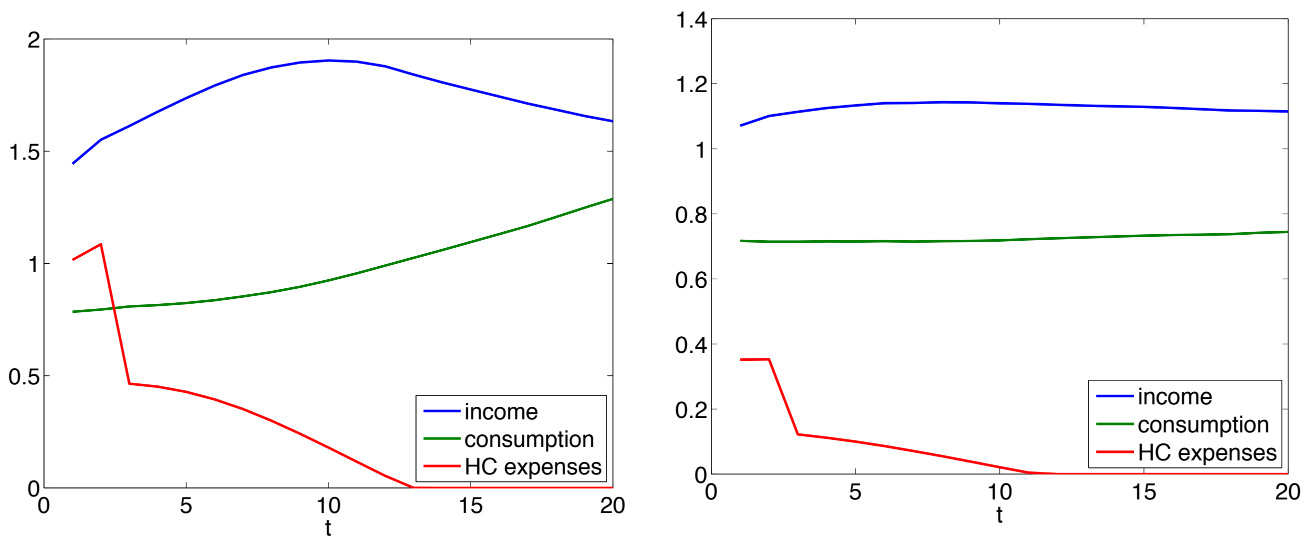
In the first stage, given the dissuasive taxes on choices which never arise in the planner's problem, the agent can either choose the full allocation (y_t and e_t) destined for some type θ_t (that could be his own true type), which will then lead him to choose also the continuation wealth optimal for that same type, or he could choose y_t optimal for some type θ_t^1 but e_t optimal for some type θ_t^2. This will however leave him with lower value given the taxes on the off-equilibrium paths. Hence, if a type deviates, he must deviate to the full allocation of another type. By the temporal incentive compatibility constraint, he will choose not to do so.

B Additional Figures and Numerical Results

B.1 Baseline economy life cycle

Figure 1: Allocations

(a) $\rho_{\theta s} = 0.2$ (b) $\rho_{\theta s} = 1.2$



Panel (a) shows the allocations of consumption, human capital expenses and income for the case $\rho_{\theta s} = 0.2$, while panel (b) shows the allocations for $\rho_{\theta s} = 1.2$.

B.2 Optimal Policies

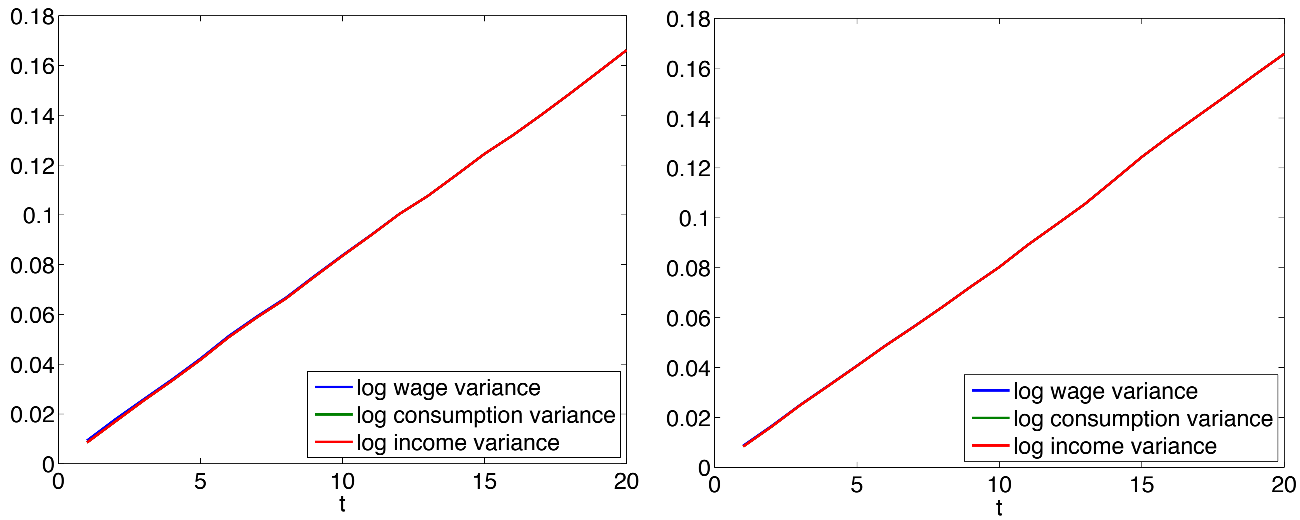
B.2.1 Properties of the Allocations

Allocations and Insurance: Figure **??** plots the average allocations over time. Average human capital investments are almost flat and highest early in life, before declining with age. Mean consumption is constant, a result due to the Inverse Euler equation in (33) , and log utility with $\beta = \frac{1}{R}$, which imply that consumption is a martingale: $E_{t-1}(c_t) = c_{t-1}$. Mean output is increasing, despite the rising labor wedge, because of the growing productivity of agents driven by their endogenous human capital investments.

Figure **??** panel (a) shows that lifetime spending on human capital is more tightly linked to lifetime income when human capital disproportionately benefits high ability agents. The causality goes both ways: higher ability agents both acquire more human capital and earn more. In turn, human capital increases earnings even further. When $\rho_{\theta s} > 1$, both effects are amplified. However, the fact that higher ability people also acquire more human capital does not mean that there is no insurance. Panel (b) highlights the extent of lifecycle insurance at the optimum by plotting the net present value of lifetime spending (consumption plus human capital expenses) against the net present value of lifetime income. Clockwise pivots of the line represent higher insurance.

Figure 2: Log variances of wages, income and consumption

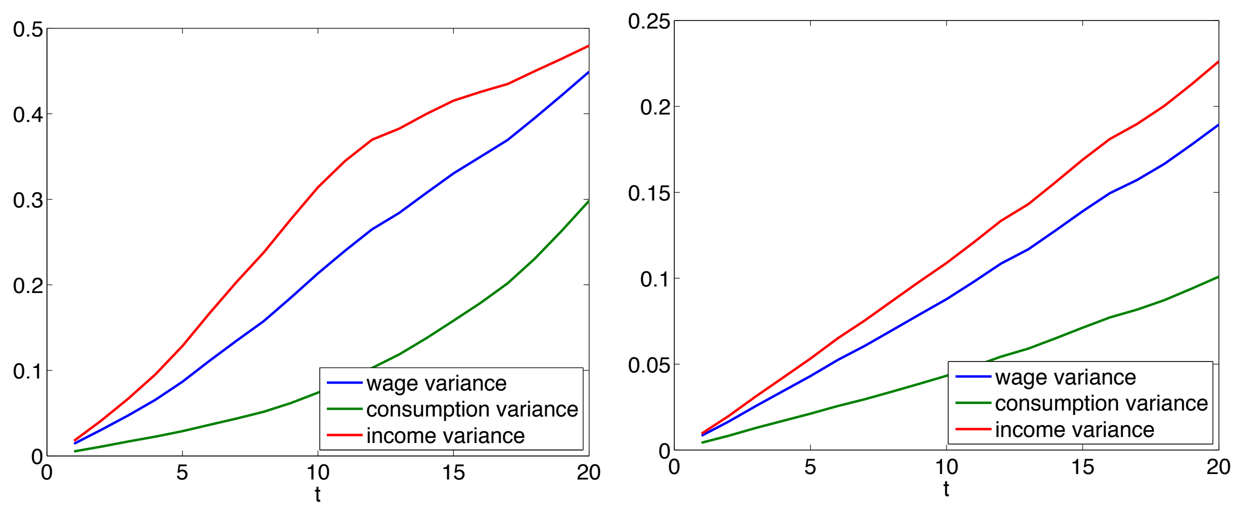
(a) $\rho_{\theta s} = 0.2$ (b) $\rho_{\theta s} = 1.2$



Panel (a) shows the variance of the logs of consumption, wage and income for the case $\rho_{\theta s} = 0.2$, while panel (b) shows the variances for $\rho_{\theta s} = 1.2$.

Figure 3: Variances of wages, income, and consumption

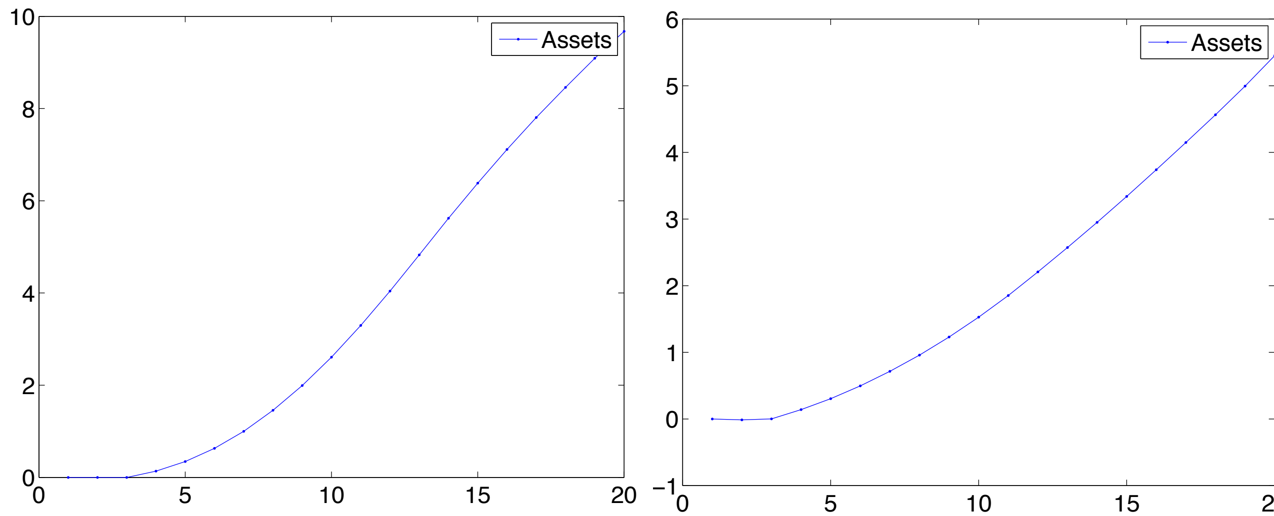
(a) $\rho_{\theta s} = 0.2$ (b) $\rho_{\theta s} = 1.2$



Panel (a) shows the variance of consumption, wage and income for the case $\rho_{\theta s} = 0.2$, while panel (b) shows the variances for $\rho_{\theta s} = 1.2$.

Figure 4: Asset holdings

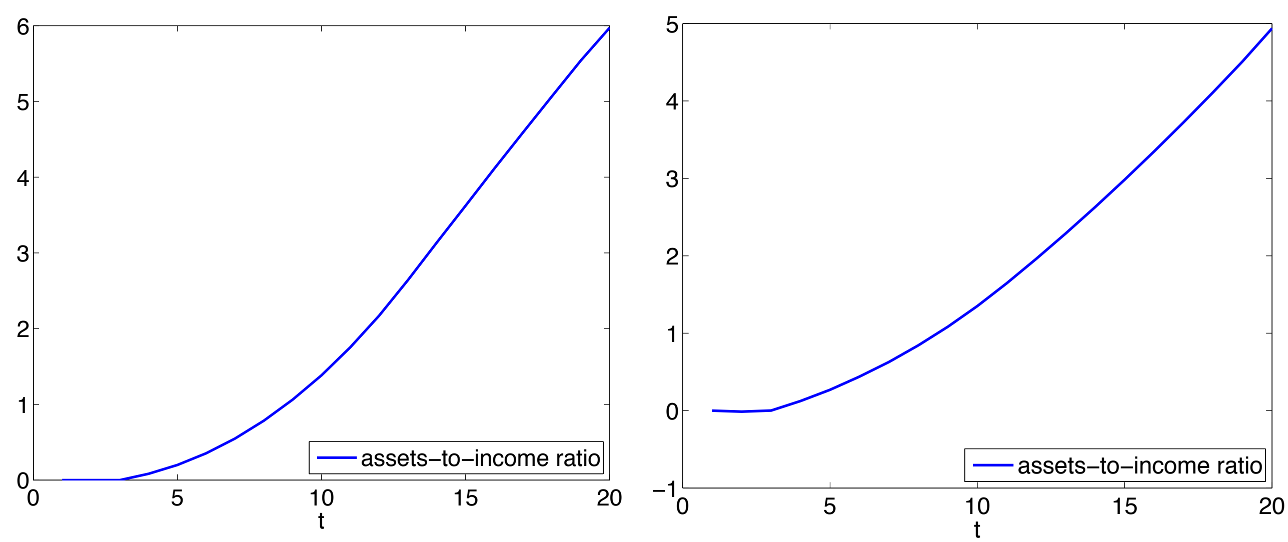
(a) $\rho_{\theta s} = 0.2$ (b) $\rho_{\theta s} = 1.2$



Panel (a) shows the mean assets accumulated by agents for the case $\rho_{\theta s} = 0.2$, while panel (b) shows the assets for $\rho_{\theta s} = 1.2$.

Figure 5: Mean assets-to-income ratio

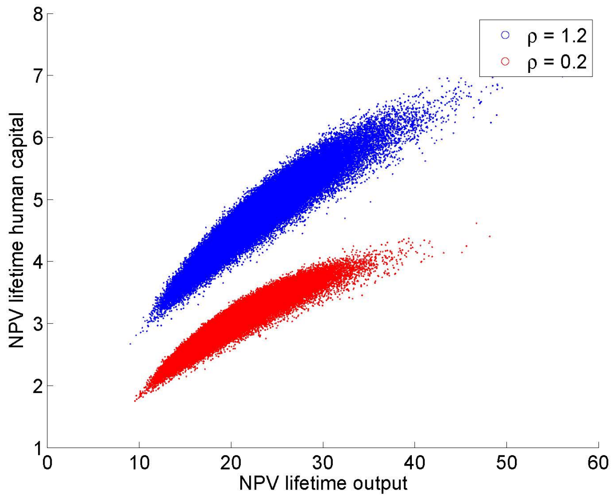
(a) $\rho_{\theta s} = 0.2$ (b) $\rho_{\theta s} = 1.2$



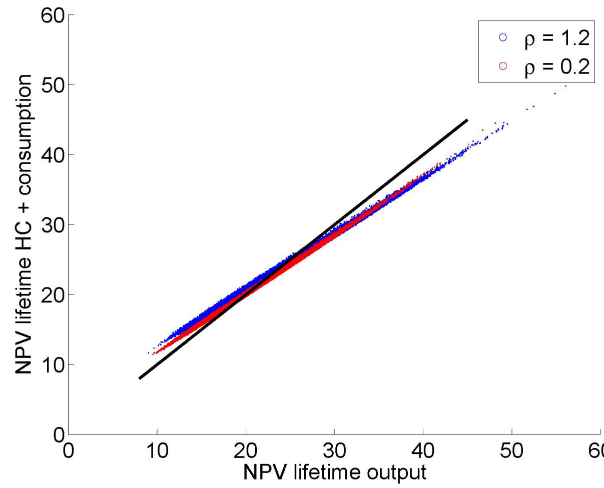
Panel (a) shows the mean assets-to-income ratio for the case $\rho_{\theta s} = 0.2$, while panel (b) shows the ratio for $\rho_{\theta s} = 1.2$.

Figure 6: Human Capital and insurance over the life cycle

(a) Human capital expenses and lifetime income



(b) Consumption and education expenses

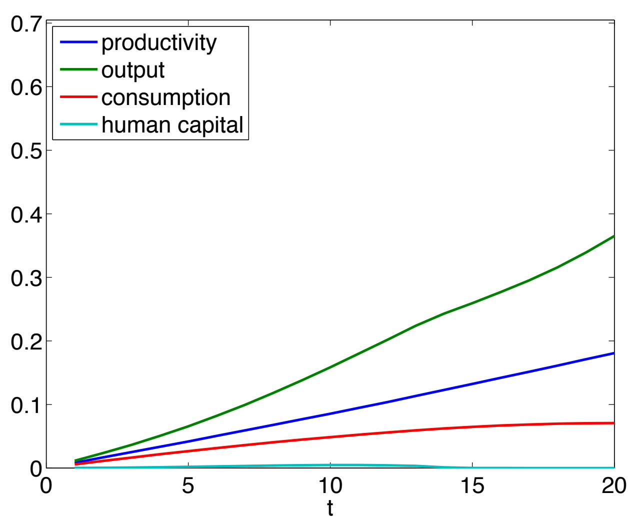


(a) Lifetime income is positively correlated with lifetime human capital expenses, the more so when $\rho_{\theta s} > 1$. There is a two-way causality: Higher ability people both acquire more human capital and have higher earnings potential. At the same time, human capital increases earnings.

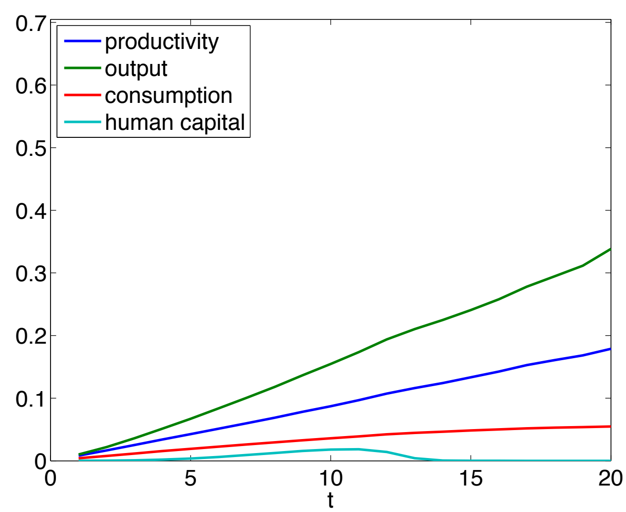
(b) The figure shows the present value of consumption and human capital expenses against the present value of lifetime income. The laissez-faire outcome is represented by the 45 degree line. Clockwise pivots of the line represent more insurance.

Figure 7: Variance and Risk

(a) $\rho_{\theta s} = 0.2$



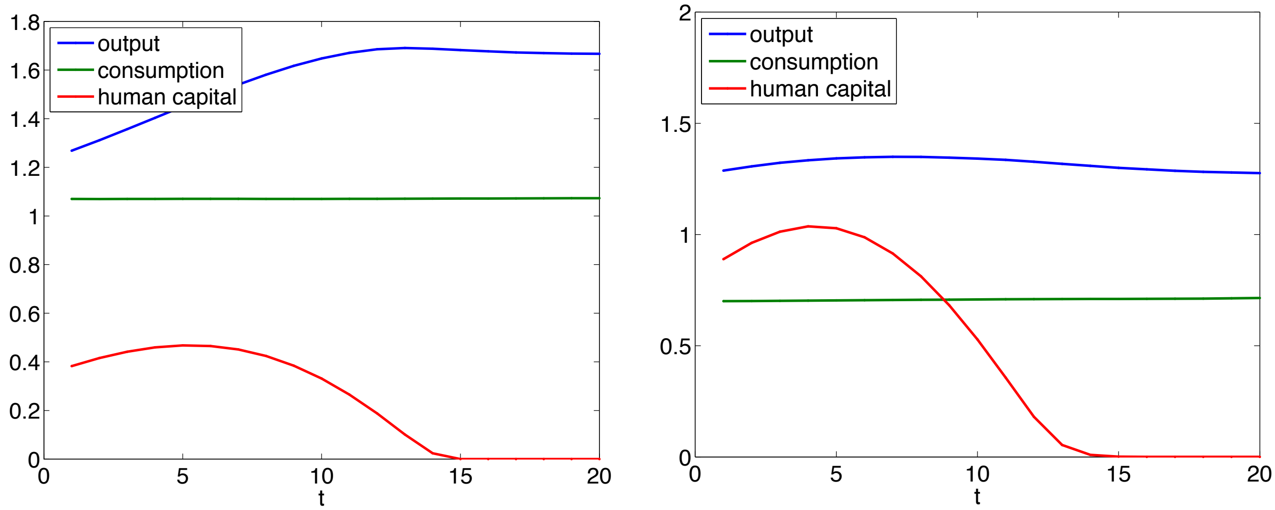
(b) $\rho_{\theta s} = 1.2$



The figure shows cross-sectional variances over time. Output is more volatile than consumption. Its volatility grows at an increasing rate, driven both by ability shocks and differential investments in human capital. But pre-tax income inequality does not fully translate into consumption inequality. All outcomes are more volatile when human capital has a negative insurance value ($\rho_{\theta s} > 1$).

Figure 8: Allocations

(a) $\rho_{\theta s} = 0.2$ (b) $\rho_{\theta s} = 1.2$



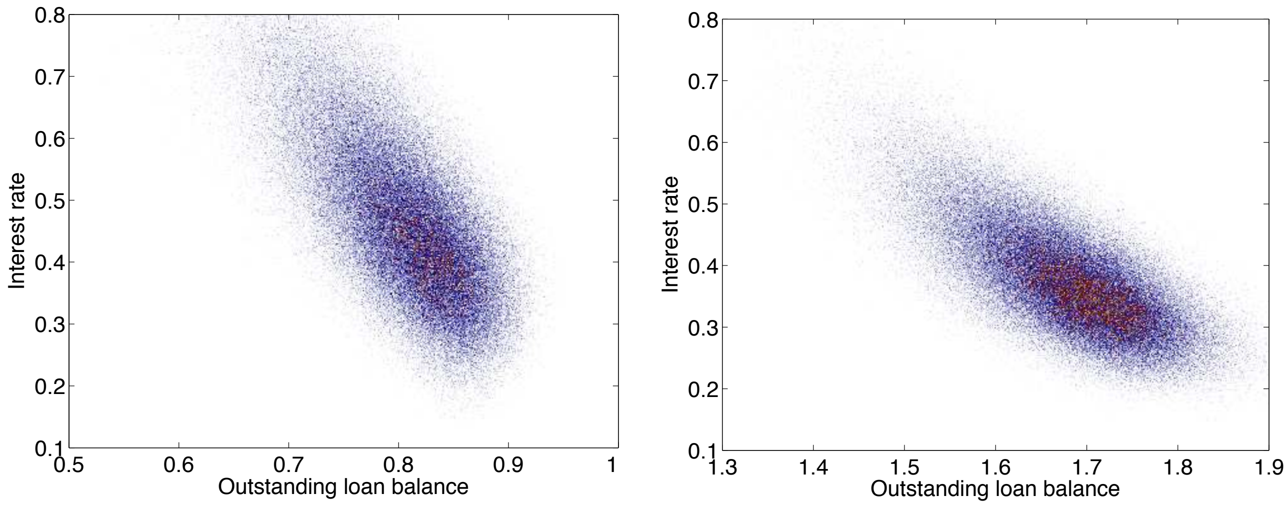
Agents optimally invest in human capital early in life. Consumption is a martingale; hence, average consumption is perfectly flat. Output and consumption are higher when human capital disproportionately benefits higher ability agents.

Finally, figure **??** describes the cross-sectional variances of output, human capital, consumption, and ability over time. The variance of output is driven not only by stochastic ability, but also by differential investments in human capital at different ability levels. Output is much more volatile than consumption. Hence, pre-tax income inequality grows at an increasing rate, but the provision of insurance prevents this from translating fully into consumption inequality. In addition, while consumption variance grows, it does so at a decreasing rate, echoing the tax and subsidy smoothing results described above.

B.2.2 Income Contingent Loan Implementation

Figure 9: Interest Rate againt the Outstanding Loan Balance

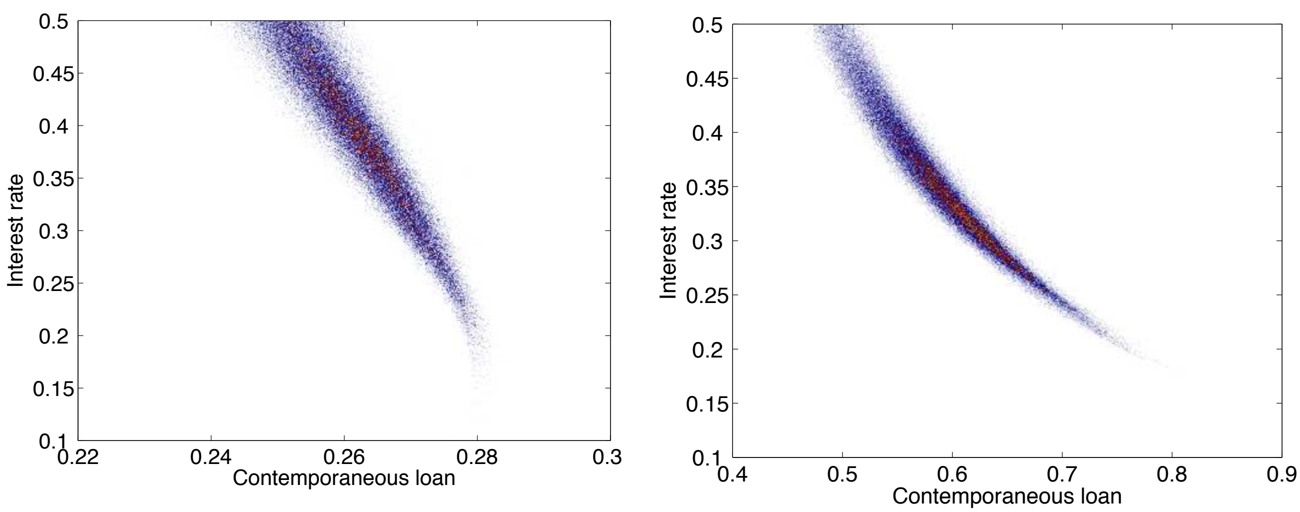
(a) $\rho_{\theta s} = 0.2$ (b) $\rho_{\theta s} = 1.2$



This figure plots a snapshot in period $t = 5$ of the implied interest rate against the Outstanding Loan Balance (each dot represents one agent, i.e., one of the 100,000 Monte Carlo simulations). Panel (a) plots the results for $\rho_{\theta s} = 0.2$ while panel (b) plots the results for $\rho_{\theta s} = 1.2$.

Figure 10: Interest Rate againt the Contemporaneous Loan

(a) $\rho_{\theta s} = 0.2$ (b) $\rho_{\theta s} = 1.2$



This figure plots a snapshot in period $t = 5$ of the implied interest rate against the contemporaneous loan given (each dot represents one agent, i.e., one of the 100,000 Monte Carlo simulations). Panel (a) plots the results for $\rho_{\theta s} = 0.2$ while panel (b) plots the results for $\rho_{\theta s} = 1.2$.