Appendix

This Appendix contains proofs of results stated in the main text, as well as auxiliary results.

A Proofs of Lemma 1 and Propositions 1 and 2

A.1 Proof of Proposition 1

Fix a continuation value function V^i for agent i, given by

$$V^i(z^i, Z) = u^i(Z) + \left(\beta_0 + \beta_1 \bar{Z}\right)\left(z^i - \bar{Z}\right) - K\left(z^i - \bar{Z}\right)^2. \tag{40}$$

In equilibrium, agent i achieves the value

$$\sup_{\hat{z}} \ \mathbb{E}\left[V^i(z_0^i + Y^i(\hat{z}), Z) + T_\kappa^i(\hat{z}, Z) \,|\, \mathcal{F}^i\right]. \tag{41}$$

Fix reports $\hat{z}^j = z_0^j$ for $j \neq i$. Substituting (40) into (41), the quantity inside the expectation of (41) is

$$\begin{aligned} &u^i(Z) + \left(\beta_0 + \beta_1 \bar{Z}\right)\left(z_0^i + Y^i(\hat{z}) - \bar{Z}\right) - K\left(z_0^i + Y^i(\hat{z}) - \bar{Z}\right)^2 \\ &\quad + \kappa_0 \left(n\kappa_2(Z) + \sum_{j=1}^n \hat{z}^j\right)^2 + \kappa_1(Z)(\hat{z}^i + \kappa_2(Z)) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2}. \end{aligned} \tag{42}$$

We can write

$$Y^i(\hat{z}) = \frac{\sum_{j=1}^n \hat{z}^j}{n} - \hat{z}^i = \frac{Z - z_0^i}{n} - \frac{n-1}{n}\hat{z}^i,$$

The terms in (42) that depend on $\hat{z}^i$ sum to

$$\left(\beta_0 + \beta_1 \bar{Z}\right)\left(-\frac{n-1}{n}\hat{z}^i\right) - K\left(\frac{n-1}{n}\right)^2 \left(z_0^i - \hat{z}^i\right)^2 + \kappa_0\left(n\kappa_2(Z) + Z - z_0^i + \hat{z}^i\right)^2 + \kappa_1(Z)\hat{z}^i.$$

The first derivative of this expression with respect to $\hat{z}^i$ is

$$\left(\beta_0 + \beta_1 \bar{Z}\right)\left(-\frac{n-1}{n}\right) + 2K\left(\frac{n-1}{n}\right)^2 \left(z_0^i - \hat{z}^i\right) + 2\kappa_0(n\kappa_2(Z) + Z - z_0^i + \hat{z}^i) + \kappa_1(Z).$$

The second derivative of (42) with respect to $\hat{z}^i$ is negative because $K > 0$ and $\kappa_0 < 0$. It follows the unique solution of this first order condition is the unique optimal report. Substituting $\hat{z}^i$ with $\hat{z}^i = z_0^i$ in the first derivative, and then equating the result to 0 implies that

$$0 = \left(\beta_0 + \beta_1 \bar{Z}\right)\left(-\frac{n-1}{n}\right) + 2\kappa_0(n\kappa_2(Z) + Z) + \kappa_1(Z),$$

and thus, for any fixed κ_1, κ_0, we have that

$$\kappa_2(Z) = -\overline{Z} + \frac{-\kappa_1(Z) + (\frac{n-1}{n})\left(\beta_0 + \beta_1 \overline{Z}\right)}{2\kappa_0 n} \tag{43}$$

is the unique $\kappa_2(Z)$ such that agent i optimally reports $\hat{z}^i = z_0^i$. This reporting strategy therefore constitutes an ex-post equilibrium of the mechanism game. Because this applies to all agents, we have

$$\frac{\sum_j \hat{z}^j}{n} - \hat{z}^i = -\left(z_0^i - \bar{Z}\right).$$

Thus, $z_0^i + Y^i(\hat{z}) = \bar{Z}$, as desired.

For the special case in which

$$\kappa_0 = \frac{-K(n-1)}{n^2},$$

we can define $Q \equiv \sum_{j \neq i} \hat{z}^j / n$ and calculate that

$$\begin{aligned}
\kappa_0 \left(\sum_j \hat{z}^j\right)^2 - K\left(z_0^i + Y^i((\hat{z}^i, \hat{z}^{-i})) - \bar{Z}\right)^2 &= \kappa_0(nQ)^2 + \kappa_0(\hat{z}^i)^2 + 2\kappa_0 nQ\hat{z}^i \\
&\quad - K\left(z_0^i + Q - \bar{Z}\right)^2 - K\left(\frac{n-1}{n}\right)^2 (\hat{z}^i)^2 \\
&\quad + 2K\frac{n-1}{n}\hat{z}^i\left(z_0^i + Q - \bar{Z}\right) \\
&= \kappa_0(nQ)^2 + \kappa_0(\hat{z}^i)^2 - K\left(z_0^i + Q - \bar{Z}\right)^2 \\
&\quad - K\left(\frac{n-1}{n}\right)^2 (\hat{z}^i)^2 + 2K\frac{n-1}{n}\hat{z}^i\left(z_0^i - \bar{Z}\right).
\end{aligned}$$

It is thus clear from the first-order optimality condition that the optimal report does not depend on Q. In this case, $\hat{z}^i = z_0^i$ is therefore a dominant strategy.

A.2 Proof of Proposition 2

Fix a continuation value as above, and let $\kappa_1(Z) = \beta_0 + \beta_1 \bar{Z}$. We see that

$$\kappa_2(Z) = -\overline{Z} - \frac{\kappa_1(Z)}{2\kappa_0 n^2}, \tag{44}$$

and thus the transfer to trader i is

$$
\begin{aligned}
&\kappa_0 \left(n\kappa_2(Z) + \sum_{j=1}^{n} \hat{z}^j \right)^2 + \kappa_1(Z)(\hat{z}^i + \kappa_2(Z)) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} \\
&= \kappa_0 \left(-Z - \frac{\kappa_1(Z)}{2\kappa_0 n} + Z \right)^2 + \kappa_1(Z)(z_0^i - \overline{Z} - \frac{\kappa_1(Z)}{2\kappa_0 n^2}) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} \\
&= \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} + \kappa_1(Z)(z_0^i - \bar{Z}) - \frac{\kappa_1^2(Z)}{2\kappa_0 n^2} + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} \\
&= \kappa_1(Z) \left(z_0^i - \bar{Z} \right) \\
&= \left(\beta_0 + \beta_1 \bar{Z} \right) \left(z_0^i - \bar{Z} \right).
\end{aligned}
$$

From Proposition 1, agent i receives the post reallocation inventory $\bar{Z}$ in equilibrium. The equilibrium utility of agent i is then simply

$$
u^i(Z) + \kappa_1(Z) \left(z_0^i - \bar{Z} \right) = u^i(Z) + \left(\beta_0 + \beta_1 \bar{Z} \right) \left(z_0^i - \bar{Z} \right).
$$

Comparing this with $V^i(z_0^i, Z)$, the result follows from the fact that $K > 0$.

For the uniqueness of $\kappa_1(\cdot)$, note that for IR to hold with probability 1, by continuity, it must hold in the event that $z_0^i = \bar{Z}$ for all i. In this case, the change in utility for each trader is just the transfer they receive. By the definition of the transfers, straightforward algebra shows that for any vector of reports,

$$
\begin{aligned}
\sum_i T_\kappa^i(\hat{z}, Z) &= \sum_i \left(\kappa_0 \left(n\kappa_2(Z) + \sum_{j=1}^{n} \hat{z}^j \right)^2 + \kappa_1(Z)(\hat{z}^i + \kappa_2(Z)) + \frac{\kappa_1^2(Z)}{4\kappa_0 n^2} \right) \\
&= -n \left(\sqrt{-\kappa_0} \left(n\kappa_2(Z) + \sum_{j=1}^{n} \hat{z}^j \right) - \frac{\kappa_1(Z)}{2\sqrt{-\kappa_0} n} \right)^2.
\end{aligned}
$$

Plugging in the κ_2 of proposition 1 and $\hat{z}^i = z_0^i$, this equals

$$
-n \left(\sqrt{-\kappa_0} \frac{-\kappa_1(Z) + (\frac{n-1}{n}) \left(\beta_0 + \beta_1 \overline{Z} \right)}{2\kappa_0} - \frac{\kappa_1(Z)}{2\sqrt{-\kappa_0} n} \right)^2,
$$

which is nonnegative if and only if $\kappa_1(Z) = \beta_0 + \beta_1 \overline{Z}$, completing the proof.

A.3 Proof of Lemma 1

Because $b \neq 0$, the following are equivalent

$$\begin{aligned} & d + \sum_{j \neq i} (a + bp + cz_t^j) = 0 \\ \iff & -b(n-1)p = d + (n-1)a + cZ_t^{-i} \\ \iff & p = \frac{-1}{b(n-1)} \left(d + (n-1)a + cZ_t^{-i} \right). \end{aligned}$$

B A lemma and the Proof of Proposition 3

First, we prove a technical lemma that will be useful in all subsequent proofs.

Lemma 2. *Let $c \neq 0$ be an arbitrary constant, and let $\bar{Z}_t$, σ_Z^2 be defined as in the text. Then, for any t,*

$$\mathbb{E}[\int_0^t e^{-cs} \bar{Z}_s ds] = \bar{Z}_0 \frac{1 - e^{-ct}}{c}, \tag{45}$$

and

$$\mathbb{E}[\left(\int_0^t e^{-cs} \bar{Z}_s ds \right)^2] = \frac{(1 - e^{-ct})^2}{c^2} \bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2} \frac{e^{-2ct} \left(2ct - 4e^{ct} + e^{2ct} + 3 \right)}{2c^3}. \tag{46}$$

If $c = 0$, then the corresponding expectations equal the limits of these expressions, and in particular

$$\mathbb{E}[\left(\int_0^t \bar{Z}_s ds \right)^2] = \bar{Z}_0^2 t^2 + \frac{\sigma_Z^2}{n^2} \frac{t^3}{3}. \tag{47}$$

Proof: Fixing s, because $\mathbb{E}[(\bar{Z}_s)^2] = \bar{Z}_0^2 + (\sigma_Z^2/n^2)s$ by assumption, we can apply Hölder's inequality to find that

$$\mathbb{E}[|e^{-cs} \bar{Z}_s|] \leq e^{-cs} \sqrt{\mathbb{E}[(\bar{Z}_s)^2]} = e^{-cs} \sqrt{\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2} s}.$$

It follows that, for any t,

$$\int_0^t \mathbb{E}[|e^{-cs} \bar{Z}_s|] ds \leq \int_0^t e^{-cs} \sqrt{\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2} s} ds < \infty.$$

We may thus apply the Fubini-Tonelli theorem to write that

$$\mathbb{E}[\int_0^t e^{-cs} \bar{Z}_s \, ds] = \int_0^t \mathbb{E}[e^{-cs} \bar{Z}_s] \, ds = \bar{Z}_0 \int_0^t e^{-cs} ds = \bar{Z}_0 \frac{1 - e^{-ct}}{c},$$

where we have used the fact that by definition of H_t, $\mathbb{E}[\bar{Z}_s] = \bar{Z}_0$. Henceforth, for brevity we refer to this as the "Hölder's inequality and Fubini-Tonelli theorem argument."

Now, define $W_t = \int_0^t e^{-cs}\bar{Z}_s\,ds$. By Ito's lemma,

$$W_t^2 = 2\int_0^t W_s e^{-cs}\bar{Z}_s ds = 2\int_0^t\int_0^s e^{-cs}\bar{Z}_s e^{-cu}\bar{Z}_u du\,ds$$

By the Lévy property, $\mathbb{E}[\bar{Z}_u(\bar{Z}_s - \bar{Z}_u)] = 0$. An application of the "Hölder's inequality and Fubini-Tonelli theorem argument" gives that

$$\begin{aligned}
\mathbb{E}[\int_0^t\int_0^s e^{-cs}\bar{Z}_s e^{-cu}\bar{Z}_u du\,ds] &= \int_0^t\int_0^s \mathbb{E}[e^{-cs}\bar{Z}_s e^{-cu}\bar{Z}_u] du\,ds \\
&= \int_0^t\int_0^s \mathbb{E}[e^{-cs}e^{-cu}(\bar{Z}_s - \bar{Z}_u + \bar{Z}_u)\bar{Z}_u] du\,ds \\
&= \int_0^t\int_0^s \mathbb{E}[e^{-cs}e^{-cu}\bar{Z}_u^2] du\,ds \\
&= \int_0^t\int_0^s e^{-cs}e^{-cu}\left(\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2}u\right)] du\,ds \\
&= \frac{(1-e^{-ct})^2}{2c^2}\bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2}\frac{e^{-2ct}\left(2ct - 4e^{ct} + e^{2ct} + 3\right)}{4c^3}.
\end{aligned}$$

Finally, starting at the penultimate line of the above system and plugging in $c = 0$, we arrive at

$$\mathbb{E}[\left(\int_0^t \bar{Z}_s ds\right)^2] = \bar{Z}_0^2 t^2 + \frac{\sigma_Z^2}{n^2}\frac{t^3}{3}. \tag{48}$$

Now, we are ready to prove proposition 3. The proof proceeds in 4 steps. First, we use admissibility to restrict the possible set of linear equilibria. Second, we show that in any linear equilibrium, the value function must take a specific linear-quadratic form. Third, we calculate the unique value function and linear coefficients consistent with the Hamilton-Jacobi Bellman (HJB) equation. Finally, we verify that the candidate value function and coefficients indeed solve the Markov control problem. Throughout, we write simply $V(z, Z)$ in place of $V^i(z, Z)$. As in the text, we let $\sigma_i^2 \equiv \mathbb{E}[(H_1^i)^2]$.

B.1 Admissibility

In this section, we show that if there were a linear equilibrium with $c \geq r/2$, then one player would be using an inadmissible strategy, meaning that the value achieved in the problem

$$V(z_0^i, Z_0) \equiv \sup_{D\in\mathcal{A}^i} \mathbb{E}\left[z_{\mathcal{T}}^D\pi - \int_0^{\mathcal{T}} \gamma\left(z_s^D\right)^2 + \Phi_{(a,b,c)}\left(D_s; Z_s - z_s^D\right)\right) D_s\,ds\right] \tag{49}$$

would be negative infinity or undefined. In order to see this, fix candidate demand coefficients (a, b, c). Then each trader demands a flow $D = a + b\phi + cz$, so the market clearing price must be

$$\phi = \frac{a + c\bar{Z}}{-b}.$$

Plugging this price back into agent demands, we can write

$$D = c(z - \bar{Z}).$$

It follows that if all agents follow this strategy, the inventory of agent i at time t is

$$z_t^i = z_0^i + c \int_0^t z_s^i - \bar{Z}_s ds + H_t^i. \tag{50}$$

Applying Ito's lemma for semimartingales to $e^{-ct} z_t^i$, and multiplying both sides by e^{ct}, one can show[29] that

$$z_t^i = e^{ct} z_0^i - e^{ct} c \int_0^t e^{-cs} \bar{Z}_s \, ds + e^{ct} \int_0^t e^{-cs} \, dH_s^i. \tag{51}$$

Because e^{-cs} is square integrable, the last term in the expression for z_t^i is a martingale, so by lemma 2 we have that

$$\mathbb{E}[z_t^i] = e^{ct} z_0^i + \bar{Z}_0(1 - e^{ct}),$$

while

$$\begin{aligned}
\mathbb{E}[(z_t^i)^2] &= \mathbb{E}[\left(e^{ct} z_0^i - e^{ct} c \int_0^t e^{-cs} \bar{Z}_s \, ds \right)^2] + e^{2ct} \mathbb{E}[(\int_0^t e^{-cs} dH_s^i)^2] \\
&= e^{2ct}(z_0^i)^2 + 2e^{ct} z_0^i \bar{Z}_0 (1 - e^{ct}) + (1 - e^{ct})^2 \bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2} \frac{(2ct - 4e^{ct} + e^{2ct} + 3)}{2c} \\
&+ e^{2ct} \mathbb{E}[(\int_0^t e^{-cs} dH_s^i)^2].
\end{aligned}$$

Applying Ito isometry for martingales, and recalling that $[H^i, H^i]_t = \sigma_i^2 t$ because H^i is square-integrable, we have

$$\mathbb{E}\left[\left(\int_0^t e^{-cs} dH_s^i \right)^2 \right] = \int_0^t e^{-2cs} \sigma_i^2 \, ds = \frac{-\sigma_i^2}{2c}(e^{-2ct} - 1).$$

Thus

$$\mathbb{E}[(z_t^i)^2] = e^{2ct}(z_0^i)^2 + 2e^{ct} z_0^i \bar{Z}_0 (1 - e^{ct}) + (1 - e^{ct})^2 \bar{Z}_0^2 + \frac{\sigma_Z^2}{n^2} \frac{(2ct - 4e^{ct} + e^{2ct} + 3)}{2c} + \frac{\sigma^i}{2c}(e^{2ct} - 1). \tag{52}$$

Applying the independence of $\mathcal{T}, H_t^i$ and Tonelli's theorem, we have

$$\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2 \, ds \right] = \int_0^\infty r e^{-rt} \int_0^t \mathbb{E}[(z_s^i)^2] \, ds \, dt \le \int_0^\infty \int_0^t \mathbb{E}\left[r e^{-rs} (z_s^i)^2 \right] \, ds \, dt.$$

From (52), we see that this quantity is finite if and only if $2c < r$. In this case, it is straight-

[29]This is exactly the derivation of the solution of the Ornstein-Uhlenbeck process.

forward to show that the quantity in (49) is finite, with

$$D = c\left(z - \bar{Z}\right).$$

B.2 Value function in a linear quadratic equilibrium

Fix demand coefficients (a, b, c) such that $c < r/2$ and $b \neq 0$. Agent i demands assets at the rate $D_t = a + b\phi_t + cz_t^i$, so the market clearing price must be

$$\phi_t = \frac{a + c\bar{Z}_t}{-b}.$$

Plugging this price back into the demand function of agent i, we can write $D_t = c(z_t^i - \bar{Z}_t)$. Because all traders follow this strategy, the inventory of agent i at time t is

$$z_t^i = z_0^i + c\int_0^t \left(z_s^i - \bar{Z}_t\right)\, ds + H_t^i. \tag{53}$$

Keeping the coefficients (a, b, c) fixed, we will now prove that in any symmetric affine equilibrium, the value function

$$V(z_0^i, Z_0) \equiv \sup_{D \in \mathcal{A}^i} \mathbb{E}\left[z_{\mathcal{T}}^D \pi - \int_0^{\mathcal{T}} \gamma\left(z_s^D\right)^2 + \Phi_{(a,b,c)}\left(D_s; Z_s - z_s^D\right)\right) D_s\, ds \right] \tag{54}$$

takes the form

$$V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z\bar{Z},$$

where

$$\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r - 2c} \\
\alpha_5 &= \frac{1}{r - c}\left(\frac{c^2}{b} - 2\alpha_3 c\right) \\
\alpha_4 &= \frac{1}{r}\left(\frac{c^2}{-b} - c\alpha_5\right) \\
\alpha_1 &= \frac{1}{r - c}\left(rv + \frac{ac}{b}\right) \\
\alpha_2 &= \frac{1}{r}\left(\frac{ca}{-b} - c\alpha_1\right) \\
\alpha_0^i &= \frac{1}{r}\left(\alpha_3 \sigma_i^2 + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_5 \frac{\rho^i}{n}\right).
\end{aligned}$$

Given the α coefficients, we have

$$
\begin{aligned}
& r\left(\alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z\bar{Z}\right) \\
&\quad = rvz - \gamma z^2 + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} - c(z-\bar{Z})\frac{a + c\bar{Z}}{-b} \\
&\qquad + c(z-\bar{Z})(\alpha_1 + 2\alpha_3 z + \alpha_5 \bar{Z})
\end{aligned}
$$

Let $Y_t = 1_{\{\mathcal{T} \le t\}}$ and $V(z, Z)$ be defined as above. Let

$$
X = \begin{bmatrix} z_t^i \\ Z_t \\ Y_t \end{bmatrix}
$$

and $U(X) = U(z, Z, Y) = (1-Y)V(z, Z) + Yvz$. Then, by Ito's lemma for semimartingales, for any t, we have

$$
\begin{aligned}
U(X_t) - U(X_0) = & \int_{0+}^t (1-Y_{s-})V_z(z_{s-}^i, Z_{s-}) + Y_{s-} v \, dz_s^i + \int_{0+}^t (1-Y_{s-})V_Z(z_{s-}^i, Z_{s-}) \, dZ_s && (55) \\
& + \frac{1}{2}\int_{0+}^t (1-Y_{s-})V_{zz}(z_{s-}^i) \, d[z^i, z^i]_s^c + \frac{1}{2}\int_{0+}^t (1-Y_{s-})V_{ZZ}(z_{s-}^i) \, d[Z, Z]_s^c && (56) \\
& + \int_{0+}^t (1-Y_{s-})V_{zZ}(z_{s-}^i) \, d[z^i, Z]_s^c && (57) \\
& + \sum_{0 \le s \le t} U(X_s) - U(X_{s-}) - [(1-Y_{s-})V_z(z_{s-}^i, Z_s) + Y_{s-} v]\Delta z_s^i && (58) \\
& - \sum_{0 \le s \le t} (1-Y_{s-})V_Z(z_{s-}^i, Z_s)\Delta Z_s, && (59)
\end{aligned}
$$

where we have used the fact that

$$
\int_{0+}^t \frac{\partial}{\partial Y} U(z_{s-}^i, Y_{s-}) \, dY_s = \sum_{0 \le s \le t} \frac{\partial}{\partial Y} U(z_{s-}^i, Y_{s-})\Delta Y_s,
$$

and the fact that $[z^i, Y]^c = [Z, Y]^c = [Y, Y]^c = 0$.

Now, we note that

$$
\begin{aligned}
V(z_s^i, Z_s) - V(z_{s-}^i, Z_{s-}) = & \alpha_1 \Delta z_s^i + \alpha_2 \frac{\Delta Z_s}{n} + \alpha_4 \left(\frac{\Delta Z_s}{n}\right)^2 + 2\alpha_4 \frac{Z_{s-}\Delta Z_s}{n^2} \\
& + \alpha_3 (\Delta z_s^i)^2 + 2\alpha_3 z_{s-}^i \Delta z_s^i + \alpha_5 z_s^i \frac{\Delta Z_s}{n} \\
& + \alpha_5 \bar{Z}_{s-}\Delta z_s^i + \alpha_5 \frac{\Delta Z_s}{n}\Delta z_s^i,
\end{aligned}
$$

while

$$
V_Z(z_{s-}^i, Z_{s-})\Delta Z_s = \frac{\Delta Z_s}{n}\left(\alpha_2 + \alpha_5 z_{s-}^i + 2\alpha_4 \bar{Z}_{s-}\right)
$$

$$V_z(z^i_{s-}, Z_{s-})\Delta z^i_s = \Delta z^i_s \left(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}\right).$$

Thus, the total contribution to the sum in (55) from jumps in z^i_s or Z_s is given by

$$(1 - Y_{s-}) \left(\alpha_4 \left(\frac{\Delta Z_s}{n} \right)^2 + \alpha_3 (\Delta z^i_s)^2 + \alpha_5 \frac{\Delta Z_s}{n} \Delta z^i_s \right)$$

because the term $-Y_{s-} v \Delta z^i_s$ is cancelled by the same term in $U(X_s) - U(X_{s-})$.

We note that jumps in z^i arise from jumps in H^i. We can thus write the sum as

$$\begin{aligned}
&\sum_{0 \le s \le t} \Delta Y_s \left(v z^i_{s-} - V(z^i_{s-}, Z_{s-}) \right) \\
&+ (1 - Y_{s-}) \left(\alpha_4 \left(\frac{\Delta Z_s}{n} \right)^2 + \alpha_3 (\Delta H^i_s)^2 + \alpha_5 \frac{\Delta Z_s}{n} \Delta H^i_s \right).
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
\int_{0+}^t V_z(z^i_{s-}, Z_{s-})\, dz^i_s &= \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})\, dz^i_s \\
&= \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \left(c(z^i_s - \bar{Z}_s) \right) ds \\
&\quad + \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})\, dH^i_s.
\end{aligned}$$

We let

$$\begin{aligned}
\chi_s = {} & c(z^i_s - \bar{Z}_s)(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} \\
& + r(v z^i_s - V(z^i_s, Z_s)).
\end{aligned}$$

Plugging in $V_{ZZ} = 2\alpha_4/n^2$, $V_{zz} = 2\alpha_3$, $V_{zZ} = \alpha_5/n$, and evaluating (55) at $t = \mathcal{T}$, we can write

$$
\begin{aligned}
U(X_{\mathcal{T}}) - U(X_0) = &\int_{0+}^{\mathcal{T}} \chi_s \, ds &(60)\\
&+ \int_{0+}^{\mathcal{T}} (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \, dH^i_s &(61)\\
&+ \int_{0+}^{\mathcal{T}} \frac{1}{n} \left(\alpha_2 + \alpha_5 z^i_{s-} + 2\alpha_4 \bar{Z}_{s-}\right) \, dZ_s &(62)\\
&+ \alpha_3 \left(-\sigma_i^2 \mathcal{T} + \int_{0+}^{\mathcal{T}} d[H^i, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta H^i_s)^2 \right) &(63)\\
&+ \frac{\alpha_4}{n^2} \left(-\sigma_Z^2 \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, Z]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s)^2 \right) &(64)\\
&+ \frac{\alpha_5}{n} \left(-\rho^i \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s \Delta H^i_s) \right) &(65)\\
&+ \int_{0}^{\mathcal{T}} \left(v z^i_{s-} - V(z^i_{s-}, Z_{s-})\right) \, (dY_s - r \, ds), &(66)
\end{aligned}
$$

where we have replaced replaced $Y_{s-} = 0$ for $s \le \mathcal{T}$, by definition. Also, we have used the fact that $[z^i, z^i]^c = [H^i, H^i]^c$ and $[z^i, Z]^c = [H^i, Z]^c$, since z^i is the sum of H^i_t and a finite variation process, where finite variation processes are quadratic pure jump semimartingales (Protter (2004)).

For any deterministic $\mathcal{T}$, it is well known from the theory of Lévy processes that

$$
\begin{aligned}
&\mathbb{E}[\left(-\sigma_i^2 \mathcal{T} + [H^i, H^i]^c_{\mathcal{T}} + \sum_{0 \le s \le \mathcal{T}} (\Delta H^i_s)^2 \right)]\\
&= \mathbb{E}[\left(-\sigma_Z^2 \mathcal{T} + [Z, Z]^c_{\mathcal{T}} + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s)^2 \right)]\\
&= \mathbb{E}[\left(-\rho^i \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s \Delta H^i_s) \right)]\\
&= 0,
\end{aligned}
$$

and since $\mathcal{T}$ is independent of Z, H^i, we may apply law of iterated expectations (conditioning on $\mathcal{T}$) to show these are still zero for exponentially distributed $\mathcal{T}$.

Now, we let $\mathcal{G}^i_\infty$ be the sigma algebra generated by the path of $\{H^i_t, Z_t\}_{t=0}^\infty$, which is independent of $\mathcal{T}$ by assumption. Then

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\mathcal{T}} [vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})]\,(dY_s - rds)\right] &= \mathbb{E}\left[\mathbb{E}\left[\int_0^{\mathcal{T}} [vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})]\,(dY_s - rds) \;\middle|\; \mathcal{G}^i_\infty\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[-r\int_0^{\mathcal{T}} [vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})]\,ds + vz^i_T - V(z^i_T, Z^i_T) \;\middle|\; \mathcal{G}^i_\infty\right]\right] \\
&= \mathbb{E}\left[-r\int_0^\infty re^{-rt}\left(\int_0^t [vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})]\,ds\right)\,dt\right] \\
&\quad + \mathbb{E}\left[\int_0^\infty re^{-rt}\left(vz^i_t - V(z^i_t, Z^i_t)\right)\,dt\right] \\
&= \mathbb{E}\left[-r\int_0^\infty \left(vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})\right)\int_s^\infty re^{-rt}\,dt\,ds\right] \\
&\quad + \mathbb{E}\left[\int_0^\infty re^{-rt}\left(vz^i_t - V(z^i_t, Z^i_t)\right)\,dt\right] \\
&= \mathbb{E}\left[-\int_0^\infty \left(vz^i_{s-} - V(z^i_{s-}, Z^i_{s-})\right)re^{-rs}\,ds\right] \\
&\quad + \mathbb{E}\left[\int_0^\infty re^{-rt}\left(vz^i_t - V(z^i_t, Z^i_t)\right)\,dt\right] = 0,
\end{aligned}$$

where the fourth equality is a change of order of integration from $\int_0^\infty \int_0^t \,ds\,dt$ to $\int_0^\infty \int_s^\infty \,dt\,ds$. Finally, we have already shown that $\mathbb{E}[(z^i_s)^2], \mathbb{E}[(z^i_s)], \mathbb{E}[(\bar{Z}_s)^2], \mathbb{E}[(\bar{Z}_s)]$ are all integrable (i.e., $\mathcal{L}^1$) processes. It then follows from Hölder's inequality that $\mathbb{E}[z^i_s \bar{Z}_s]$ is also integrable. Then $(\alpha_1 + \alpha_5\bar{Z}_s + 2\alpha_3 z^i_s)$ and $\left(\alpha_2 + \alpha_5 z^i_s + 2\alpha_4\bar{Z}_s\right)$ are square integrable processes, so for fixed $\mathcal{T}$,

$$\begin{aligned}
&\mathbb{E}[\int_{0+}^{\mathcal{T}} (\alpha_1 + \alpha_5\bar{Z}_{s-} + 2\alpha_3 z^i_{s-})\,dH^i_s] \\
&= \mathbb{E}[\int_{0+}^{\mathcal{T}} \frac{1}{n}\left(\alpha_2 + \alpha_5 z^i_{s-} + 2\alpha_4\bar{Z}_{s-}\right)\,dZ_s] = 0,
\end{aligned}$$

since H^i, Z are martingales. Applying law of iterated expectations conditioning on $\mathcal{T}$, the same is true by independence when $\mathcal{T}$ is exponentially distributed. We have thus shown that taking an expectation in equation (60) reduces to

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}[\int_{0+}^{\mathcal{T}} \chi_s\,ds]. \tag{67}$$

Because α^i_0 through α_5 satisfy the system of equations specified at the beginning of this proof, we have

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left[\int_{0+}^{\mathcal{T}} c(z^i_s - \bar{Z}_s)\frac{a + c\bar{Z}_s}{-b} + \gamma(z^i_s)^2\,ds\right].$$

Note that, by definition, $\mathbb{E}[U(X_{\mathcal{T}})] = \mathbb{E}[vz^i_{\mathcal{T}}] = \mathbb{E}[\pi z^i_{\mathcal{T}}]$, and $\mathbb{E}[U(X_0)] = U(X_0) = V(z^i_0, Z_0)$. We can thus rearrange to find that

$$
\begin{aligned}
V(z^i_0, Z_0) &= \mathbb{E}\left[\pi z^i_{\mathcal{T}} + \int_{0+}^{\mathcal{T}} -c(z^i_s - \bar{Z}_s)\frac{a + c\bar{Z}_s}{-b} - \gamma(z^i_s)^2 ds\right] \\
&= \mathbb{E}\left[\pi z^i_{\mathcal{T}} + \int_{0+}^{\mathcal{T}} -c(z^i_s - \bar{Z}_s)\phi_t - \gamma(z^i_s)^2 ds\right],
\end{aligned}
$$

which completes the proof.

B.3 Solving the HJB equation

For conjectured demand function coefficients a, b, c, the HJB equation is

$$
\begin{aligned}
rV(z, Z) = -\gamma z^2 + vz + \frac{\sigma_i^2}{2}V_{zz}(z, Z) + \frac{\sigma_Z^2}{2}V_{ZZ}(z, Z) + \rho^i V_{zZ}(z, Z) \\
+ \sup_D -\Phi_{(a,b,c)}(D; Z - z)D + V_z(z, Z)D.
\end{aligned}
$$

Plugging in

$$
\Phi_{(a,b,c)}(D; Z - z) = \frac{-1}{b(n-1)}[D + (n-1)a + c(Z - z)]
$$

from Lemma 1 and taking a derivative with respect to D, we have

$$
\frac{1}{b(n-1)}(2D + (n-1)a + c(Z - z)) + V_z(z, Z) = 0,
$$

or

$$
D = -\frac{1}{2}[(n-1)a + c(Z - z) + b(n-1)V_z(z, Z)].
$$

From the above, in any linear equilibrium, it must be that $V_z(z, Z) = \alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z$. Then

$$
D = -\frac{1}{2}[(n-1)a + c(Z - z) + b(n-1)\left(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z\right)], \tag{68}
$$

where the second-order condition is satisfied if and only if $b < 0$. If agent i is to find the prescribed linear strategy optimal, then D must take the form $D = a + b\phi + cz$. Further, the market clearing price must be

$$
\phi = \frac{a + c\bar{Z}}{-b}.
$$

Recall from the above that

$$
\alpha_1 + \alpha_5\bar{Z} = \frac{1}{r - c}\left(rv - 2\alpha_3 c\bar{Z} - c\left(\frac{a + c\bar{Z}}{-b}\right)\right). \tag{69}
$$

Using

$$Z = n\frac{-b\phi - a}{c},$$

we have

$$\begin{aligned}
\alpha_1 + \alpha_5 \bar{Z} &= \frac{1}{r-c}\left(rv - 2\alpha_3(-b\phi - a) - c\phi\right) \\
D &= -\frac{1}{2}[(n-1)a + n(-b\phi - a) - cz + b(n-1)\left(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z\right)].
\end{aligned}$$

So, matching coefficients from D, we require that

$$\begin{aligned}
c &= \frac{1}{2}[c - 2b(n-1)\alpha_3] \\
b &= -\frac{1}{2}[-nb + b(n-1)[\frac{1}{r-c}(2\alpha_3 b - c)]] \\
a &= -\frac{1}{2}[(n-1)a - na + b(n-1)\frac{1}{r-c}(rv + 2\alpha_3 a)].
\end{aligned}$$

Cleaning up and rearranging terms,

$$c = -2b(n-1)\alpha_3 \tag{70}$$

$$(n-2)(r-c) = 2(n-1)\alpha_3 b - c(n-1). \tag{71}$$

Combining (70, 71), we see from (71) that

$$c = \frac{-(n-2)r}{2}. \tag{72}$$

Recalling from the above that

$$\alpha_3 = \frac{-\gamma}{r-2c} = \frac{-\gamma}{r(n-1)},$$

we have

$$b = \frac{-(n-2)r^2}{4\gamma}.$$

Turning to the equation for a, we use the fact that

$$\frac{1}{r-c} = \frac{2}{nr}$$

to obtain

$$
\begin{aligned}
-a &= b(n-1)\frac{2}{nr}(rv - \frac{2\gamma}{r(n-1)}a) \\
a &= \frac{(n-2)r}{2\gamma n}(n-1)\left(rv - \frac{2\gamma}{r(n-1)}a\right) \\
a\left(1 + \frac{(n-2)}{n}\right) &= \frac{(n-2)r^2 v}{2\gamma n}(n-1) \\
a &= \frac{(n-2)r^2 v}{4\gamma}.
\end{aligned}
$$

Plugging these in, we see that

$$
\phi = \frac{a + c\bar{Z}}{-b} = v - \frac{2\gamma}{r}\bar{Z}.
$$

Then returning to $\alpha_1 + \alpha_5\bar{Z}$, we see that

$$
\begin{aligned}
\alpha_1 + \alpha_5\bar{Z} &= \frac{1}{r-c}\left(rv - 2\alpha_3 c\bar{Z} - c\left(\frac{a + c\bar{Z}}{-b}\right)\right) \\
&= \frac{2}{rn}\left(rv - 2\left(\frac{-\gamma}{r(n-1)}\right)\left(\frac{-(n-2)r}{2}\right)\bar{Z} - \left(\frac{-(n-2)r}{2}\right)\left(v - \frac{2\gamma}{r}\bar{Z}\right)\right) \\
&= \frac{2}{rn}\left(\frac{nrv}{2} - \frac{\gamma(n-2)}{(n-1)}\bar{Z} - (n-2)\gamma\bar{Z}\right) \\
&= v - \frac{2\gamma}{r}\bar{Z} + \frac{2\gamma}{r(n-1)}\bar{Z}.
\end{aligned}
$$

This must hold for any $\bar{Z}$ realization, so $\alpha_1 = v$ and

$$
\alpha_5 = -\frac{2\gamma}{r} + \frac{2\gamma}{r(n-1)}.
$$

Combining this with $\frac{a}{b} = -v$ from above, we have

$$
\alpha_2 = \frac{1}{r}\left(\frac{ca}{-b} - c\alpha_1\right) = \frac{c}{r}(v - v) = 0. \tag{73}
$$

Since $c/b = 2\gamma/r$, we see that

$$
\frac{c}{b} + \alpha_5 = \frac{2\gamma}{r(n-1)},
$$

so

$$
\alpha_4 = \frac{1}{r}\left(\frac{c^2}{-b} - c\alpha_5\right) = \frac{-c}{r}\frac{2\gamma}{r(n-1)} = \frac{\gamma(n-2)}{r(n-1)}.
$$

Finally, plugging in formulas,

$$\begin{aligned}
\alpha_0^i &= \frac{1}{r}\left(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}\right) \\
&= \frac{\gamma}{r^2}\left(-\frac{1}{n-1}\sigma_i^2 + \frac{n-2}{n-1}\frac{\sigma_Z^2}{n^2} + 2(\frac{1}{n-1}-1)\frac{\rho^i}{n}\right) \\
&= \frac{\gamma\sigma_Z^2}{r^2n^2} - \frac{\gamma}{r^2(n-1)}\left(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}\right) - \frac{2\gamma\rho^i}{r^2n} = \theta_i.
\end{aligned}$$

Putting this together, we see the unique value function and demand coefficients satisfying the HJB are given by the constants $a, b, c, \alpha_0^i, \alpha_1 - \alpha_5$ shown above. Rearranging slightly,

$$\begin{aligned}
V(z, Z) &= \alpha_0^i + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z} \\
V(z, Z) &= \alpha_0^i + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z} + v\bar{Z} - v\bar{Z} + \frac{\gamma}{r}\bar{Z}^2 - \frac{\gamma}{r}\bar{Z}^2 \\
&= \theta_i + v\bar{Z} - \frac{\gamma}{r}\bar{Z}^2 + \left(v - \frac{2\gamma}{r}\bar{Z}\right)\left(z - \bar{Z}\right) - \frac{\gamma}{r(n-1)}\left(z - \bar{Z}\right)^2.
\end{aligned}$$

B.4 Finishing the verification of optimality

We have shown that in a linear equilibrium, value functions are quadratic and in particular must be twice continuously differentiable. The HJB equation of the previous subsection is thus a necessary condition. Moreover, there is a unique candidate linear equilibrium which satisfies this HJB equation. We have therefore shown that if each player follows the proposed linear strategy, the agents indeed get their candidate value functions as their continuation values. It remains to show that each agent prefers this to any other strategy.

We adopt the notation of Section (B.2). We fix the demand-function coefficients a, b, c of the previous subsection, and the corresponding constants $\alpha_0^i, \alpha_1 - \alpha_5$ for some agent i. Fix an admissible demand rate process D^i, so that the inventory of agent i at time t is

$$z_t^D = z_0^i + \int_0^t D_s^i ds + H_t^i, \tag{74}$$

and the agent's expected inventory costs are finite. Following the same steps taken in Section (B.2), we can show that

$$\mathbb{E}[U(X_\mathcal{T}) - U(X_0)] = \mathbb{E}[\int_0^\mathcal{T} (\alpha_1 + \alpha_5\bar{Z}_s + 2\alpha_3 z_s^D)D_s^i + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_3\sigma_i^2 + \alpha_5\frac{\rho^i}{n} + r[vz_s^D - V(z_s^D, Z_s)]\, ds].$$

Because the function $(z, Z) \mapsto V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z}$ satisfies the HJB equation,

$$(\alpha_1 + \alpha_5 \bar{Z}_s + 2\alpha_3 z_s^D) D_s^i + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} + r[v z_s^D - V(z_s^D, Z_s)]$$
$$\leq \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^D) D_s^i + \gamma (z_s^D)^2.$$

Thus

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] \leq \mathbb{E}[\int_0^{\mathcal{T}} \Phi_s^D D_s^i + \gamma (z_s^D)^2 ds]$$

Applying the steps of Section (B.2), it follows that

$$V(z_0^i, Z_0) \geq \mathbb{E}\left[\int_0^{\mathcal{T}} -\Phi_s^D D_s^i - \gamma (z_s^D)^2 ds + \pi z_{\mathcal{T}}^D\right].$$

From the analysis of Section (B.2), this inequality is an equality for the proposed linear strategy $D^i = c(z - \bar{Z})$. It follows this linear strategy is optimal.

C Proof of Proposition 4

The proof proceeds in five steps. First, we use admissibility and the truthtelling property to restrict the possible set of equilibria. Second, we show that in any equilibrium, the value function must take a specific linear-quadratic form. Third, we use individual rationality to restrict the possible mechanism-transfer coefficients, and characterize the optimal mechanism reports in the equilibrium. Fourth, we calculate the unique value function and linear coefficients consistent with the HJB equation. Finally, we verify that the candidate value function and these coefficients indeed solve the Markov control problem. Throughout, we write $V(z, Z)$ in place of $V^i(z, Z)$.

C.1 Efficient allocations and admissibility

Fix a symmetric equilibrium (a, b, c). First, recall that in a symmetric equilibrium, the market clearing price ϕ_t satisfies $na + nb\phi_t + cZ_t = 0$, which implies that

$$\phi_t = \frac{a + c\bar{Z}_t}{-b},$$

and thus $a + b\phi_t + cz_t^i = c(z_t^i - \bar{Z}_t)$. In equilibrium each trader reports $\hat{z}^j = z^j$, so in equilibrium, the post-mechanism allocation of agent i is

$$z_t^i + \frac{\sum_j \hat{z}_t^j}{n} - \hat{z}_t^i = \bar{Z}_t.$$

The inventory of agent i at time t is

$$z_t^i = z_0^i + c\int_0^t \left(z_s^i - \bar{Z}_s\right)\,ds + H_t^i - \int_0^t \left(z_{s-}^i - \bar{Z}_s\right)dN_s. \tag{75}$$

As in the proof of Proposition 3,

$$e^{-ct}z_t^i = z_0^i - c\int_0^t e^{-cs}\bar{Z}_s\,ds + \int_0^t e^{-cs}\,dH_s^i - \int_0^t e^{-cs}\left(z_{s-}^i - \bar{Z}_s\right)\,dN_s.$$

Letting T_1 denote the minimum of $\mathcal{T}$ and the first jump time of N, we note that

$$-\gamma\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2\,ds\right] \le -\gamma\mathbb{E}\left[\int_0^{T_1} (z_s^i)^2\,ds\right].$$

For $t < T_1$,

$$z_t^i = e^{ct}z_0^i - ce^{ct}\int_0^t e^{-cs}\bar{Z}_s\,ds + e^{ct}\int_0^t e^{-cs}\,dH_s^i.$$

So, by lemma 2 and the steps used in the proof of Proposition 3, we know that $\mathbb{E}\left[\int_0^{T_1}(z_s^i)^2\,ds\right]$ is finite if and only if $2c < r+\lambda$. This is true regardless of z_0^i. By a straightforward application of monotone convergence, as long as $2c < r+\lambda$, this implies that

$$\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2\,ds\right] = \mathbb{E}\left[\lim_{n\to\infty}\int_0^{T_n} (z_s^i)^2\,ds\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_0^{T_n} (z_s^i)^2\,ds\right] < \infty.$$

C.2 Linear-quadratic value function

Fix a symmetric equilibrium $\mathcal{C} = (a,b,c)$. As above, the market clearing price ϕ_t satisfies $na + nb\phi_t + cZ_t = 0$, which implies that

$$\phi_t = \frac{a + c\bar{Z}_t}{-b},$$

and thus $a + b\phi_t + cz_t^i = c(z_t^i - \bar{Z}_t)$.

Recall that the transfers are given by

$$\kappa_0\left(n\kappa_2(Z_t) + \sum_j \hat{z}_t^j\right)^2 + \kappa_1(Z_t)(\hat{z}_t^i + \kappa_2(Z_t)) + \frac{\kappa_1^2(Z_t)}{4\kappa_0 n^2}.$$

Plugging in the formulas for $\hat{z}^j = z^j$, we see that for any affine κ_1, κ_2 functions, this takes the form

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

for constants R_0 through R_4 that depend on $\kappa_0, \kappa_1, \kappa_2$.

We are now ready to show that, in any linear-quadratic symmetric equilibrium, the value function

$$V(z,Z) = \mathbb{E}\left[\pi z^i_{\mathcal{T}} + \int_0^{\mathcal{T}} (-\gamma(z^i_s)^2 - c(z^i_s - \bar{Z}_s)\left(\frac{a + c\bar{Z}_s}{-b}\right) ds) + \int_0^{\mathcal{T}} T^i_\kappa(\hat{z}_s, Z_s)\, dN_s\right]$$

takes the form

$$V(z,Z) = \alpha^i_0 + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z\bar{Z},$$

where

$$\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r + \lambda - 2c} \\
\alpha_5 &= \frac{1}{r + \lambda - c}(\frac{c^2}{b} - 2\alpha_3 c + \lambda n R_3) \\
\alpha_4 &= \frac{1}{r}(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2) \\
\alpha_1 &= \frac{1}{r + \lambda - c}(rv + \frac{ac}{b} + \lambda R_4) \\
\alpha_2 &= \frac{1}{r}(\frac{ca}{-b} + (\lambda - c)\alpha_1 + \lambda n R_1) \\
\alpha^i_0 &= \frac{1}{r}(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \lambda R_0),
\end{aligned}$$

and where R_0 through R_4 are the previously defined transfer coefficients. Given the α coefficients, we have

$$\begin{aligned}
&(r + \lambda)\left(\alpha^i_0 + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z}\right) \\
&\quad = rvz - \gamma z^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_3\sigma_i^2 + \alpha_5\frac{\rho^i}{n} - c(z - \bar{Z})\frac{a + c\bar{Z}}{-b} \\
&\qquad + c(z - \bar{Z})(\alpha_1 + 2\alpha_3 z + \alpha_5\bar{Z}) + \lambda(\alpha^i_0 + \alpha_1\bar{Z} + \alpha_2\bar{Z} + \alpha_3\bar{Z}^2 \\
&\qquad + \alpha_4\bar{Z}^2 + \alpha_5\bar{Z}^2 + R_0 + R_1 Z + R_2 Z^2 + R_3 Z z + R_4 z).
\end{aligned}$$

Let $Y_t = 1_{\{\mathcal{T} \le t\}}$ and $V(z,Z)$ be defined as above. Let

$$X = \begin{bmatrix} z^i_t \\ Z_t \\ Y_t \end{bmatrix}$$

and $U(X) = U(z,Z,Y) = (1 - Y)V(z,Z) + Yvz$. Then, by Ito's lemma for semimartingales, for any t, we have

$$
\begin{aligned}
U(X_t) - U(X_0) = {} & \int_{0+}^{t} (1-Y_{s-}) V_z(z^i_{s-}, Z_{s-}) + Y_{s-} v \, dz^i_s + \int_{0+}^{t} (1-Y_{s-}) V_Z(z^i_{s-}, Z_{s-}) \, dZ_s && (76)\\
& + \frac{1}{2} \int_{0+}^{t} (1-Y_{s-}) V_{zz}(z^i_{s-}) \, d[z^i, z^i]^c_s + \frac{1}{2} \int_{0+}^{t} (1-Y_{s-}) V_{ZZ}(z^i_{s-}) \, d[Z,Z]^c_s && (77)\\
& + \int_{0+}^{t} (1-Y_{s-}) V_{zZ}(z^i_{s-}) \, d[z^i, Z]^c_s && (78)\\
& + \sum_{0 \le s \le t} U(X_s) - U(X_{s-}) - [(1-Y_{s-}) V_z(z^i_{s-}, Z_s) + Y_{s-} v] \Delta z^i_s && (79)\\
& - \sum_{0 \le s \le t} (1-Y_{s-}) V_Z(z^i_{s-}, Z_s) \Delta Z_s, && (80)
\end{aligned}
$$

where we have used the fact that

$$
\int_{0+}^{t} \frac{\partial}{\partial Y} U(z^i_{s-}, Y_{s-}) \, dY_s = \sum_{0 \le s \le t} \frac{\partial}{\partial Y} U(z^i_{s-}, Y_{s-}) \Delta Y_s,
$$

and the fact that $[z^i, Y]^c = [Z, Y]^c = [Y, Y]^c = 0$.

Now, we note that

$$
\begin{aligned}
V(z^i_s, Z_s) - V(z^i_{s-}, Z_{s-}) = {} & \alpha_1 \Delta z^i_s + \alpha_2 \frac{\Delta Z_s}{n} + \alpha_4 \left(\frac{\Delta Z_s}{n} \right)^2 + 2\alpha_4 \frac{Z_{s-} \Delta Z_s}{n^2} \\
& + \alpha_3 (\Delta z^i_s)^2 + 2\alpha_3 z^i_{s-} \Delta z^i_s + \alpha_5 z^i_s \frac{\Delta Z_s}{n} \\
& + \alpha_5 \bar{Z}_{s-} \Delta z^i_s + \alpha_5 \frac{\Delta Z_s}{n} \Delta z^i_s,
\end{aligned}
$$

while

$$
\begin{aligned}
V_Z(z^i_{s-}, Z_{s-}) \Delta Z_s &= \frac{\Delta Z_s}{n} \left(\alpha_2 + \alpha_5 z^i_{s-} + 2\alpha_4 \bar{Z}_{s-} \right) \\
V_z(z^i_{s-}, Z_{s-}) \Delta z^i_s &= \Delta z^i_s \left(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-} \right).
\end{aligned}
$$

Thus, the total contribution to the sum in (76) from jumps in z^i_s or Z_s is given by

$$
(1 - Y_{s-}) \left(\alpha_4 \left(\frac{\Delta Z_s}{n} \right)^2 + \alpha_3 (\Delta z^i_s)^2 + \alpha_5 \frac{\Delta Z_s}{n} \Delta z^i_s \right)
$$

because the term $-Y_{s-} v \Delta z^i_s$ is cancelled by the same term in $U(X_s) - U(X_{s-})$.

We note that jumps in z^i arise from jumps in both H^i and N. By independence, $\Delta N \Delta H^i =$

$\Delta N \Delta Z = 0$ with probability 1. In summary, we can write the sum as

$$\sum_{0 \le s \le t} \Delta Y_s \left(v z^i_{s-} - V(z^i_{s-}, Z_{s-}) \right)$$
$$+ (1 - Y_{s-}) \left(\alpha_4 \left(\frac{\Delta Z_s}{n} \right)^2 + \alpha_3 (\Delta H^i_s)^2 + \alpha_3 \Delta N_s (z^i_{s-} - \bar{Z}_{s-})^2 + \alpha_5 \frac{\Delta Z_s}{n} \Delta H^i_s \right).$$

It will be convenient to write

$$\sum_{0 \le s \le t} (1 - Y_{s-}) \left(\alpha_3 \Delta N_s (z^i_{s-} - \bar{Z}_{s-})^2 \right) = \int_0^t (1 - Y_{s-}) \alpha_3 (z^i_{s-} - \bar{Z}_{s-})^2 \, dN_s$$
$$= \int_0^t (1 - Y_{s-}) \alpha_3 (z^i_{s-} - \bar{Z}_{s-})^2 \, (dN_s - \lambda \, ds) + \int_0^t (1 - Y_{s-}) \lambda \alpha_3 (z^i_{s-} - \bar{Z}_{s-})^2 \, ds.$$

Finally, we note that

$$\begin{aligned}
\int_{0+}^t V_z(z^i_{s-}, Z_{s-}) \, dz^i_s &= \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \, dz^i_s \\
&= \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \left((c - \lambda)(z^i_s - \bar{Z}_s) \right) \, ds \\
&\quad + \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \, dH^i_s \\
&\quad + \int_{0+}^t (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})(\bar{Z}_s - z^i_{s-}) \, d(N_s - \lambda \, ds).
\end{aligned}$$

We let

$$\begin{aligned}
\chi_s = {}& c(z^i_s - \bar{Z}_s)(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n} \\
& - \lambda (z^i_s - \bar{Z}_s)(\alpha_1 + \alpha_5 \bar{Z}_{s-} + \alpha_3 (z^i_{s-} + \bar{Z}_{s-})) + r(v z^i_s - V(z^i_s, Z_s)).
\end{aligned}$$

Plugging in $V_{ZZ} = 2\alpha_4/n^2$, $V_{zz} = 2\alpha_3$, $V_{zZ} = \alpha_5/n$, and evaluating (76) at $t = \mathcal{T}$, we can write

$$\begin{aligned}
U(X_{\mathcal{T}}) - U(X_0) &= \int_{0+}^{\mathcal{T}} \chi_s \, ds \\
&+ \int_{0+}^{\mathcal{T}} (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-}) \, dH^i_s \\
&+ \int_{0+}^{\mathcal{T}} (\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z^i_{s-})(\bar{Z}_s - z^i_s) \, d(N_s - \lambda \, ds) \\
&+ \int_{0}^{\mathcal{T}} \alpha_3 (z^i_{s-} - \bar{Z}_{s-})^2 \, (dN_s - \lambda \, ds) + \int_{0+}^{\mathcal{T}} \frac{1}{n} \left(\alpha_2 + \alpha_5 z^i_{s-} + 2\alpha_4 \bar{Z}_{s-}\right) \, dZ_s \\
&+ \alpha_3 \left(-\sigma_i^2 \mathcal{T} + \int_{0+}^{\mathcal{T}} d[H^i, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta H^i_s)^2 \right) \\
&+ \frac{\alpha_4}{n^2} \left(-\sigma_Z^2 \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, Z]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s)^2 \right) \\
&+ \frac{\alpha_5}{n} \left(-\rho^i \mathcal{T} + \int_{0+}^{\mathcal{T}} d[Z, H^i]^c_s + \sum_{0 \le s \le \mathcal{T}} (\Delta Z_s \Delta H^i_s) \right) \\
&+ \int_{0}^{\mathcal{T}} \left(v z^i_{s-} - V(z^i_{s-}, Z_{s-}) \right) \, (dY_s - r \, ds),
\end{aligned}$$

where we have replaced $Y_{s-} = 0$ for $s \le \mathcal{T}$, by definition. Since H^i, Z are finite-variance processes, we can now apply arguments similar to those used in the proof of Proposition 3 to show that

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left[\int_{0+}^{\mathcal{T}} \chi_s \, ds \right].$$

Because α_0 through α_5 satisfy the system of equations specified at the beginning of this proof, we have

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left[\int_{0+}^{\mathcal{T}} \bar{\chi}_s \, ds \right],$$

where

$$\bar{\chi}_s = c(z^i_s - \bar{Z}_s) \frac{a + c\bar{Z}_s}{-b} + \gamma (z^i_s)^2 - \lambda (R_0 + R_1 Z_s + R_2 Z_s^2 + R_3 Z_s z^i_s + R_4 z^i_s).$$

Using the definitions of $U, \mathcal{T}$, and R_0 through R_4, as well as the fact that $\mathbb{E}[v z^i_{\mathcal{T}}] = \mathbb{E}[\pi z^i_{\mathcal{T}}]$, we can rearrange to find that

$$
\begin{aligned}
V(z_0^i, Z_0) &= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0+}^{\mathcal{T}} \bar{\chi}_s \, ds\right] \\
&= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0+}^{\mathcal{T}} -c(z_s^i - \bar{Z}_s)\frac{a + c\bar{Z}_s}{-b} - \gamma(z_s^i)^2 + \lambda T_\kappa^i(\hat{z}_s, Z_s)\, ds\right] \\
&= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0}^{\mathcal{T}} -c(z_s^i - \bar{Z}_s)\frac{a + c\bar{Z}_s}{-b} - \gamma(z_s^i)^2 \, ds + \int_0^{\mathcal{T}} T_\kappa^i(\hat{z}_s, Z_s)\, dN_s\right],
\end{aligned}
$$

which completes the proof.

C.3 The Mechanism

Fix a symmetric equilibrium. Recall the mechanism transfers are given by

$$
\kappa_0 \left(n\kappa_2(Z_t) + \sum_j \hat{z}_t^j\right)^2 + \kappa_1(Z_t)(\hat{z}_t^i + \kappa_2(Z_t)) + \frac{\kappa_1^2(Z_t)}{4\kappa_0 n^2}.
$$

For the purpose of this proof, we will treat κ_1, κ_2 as arbitrary affine functions, and show the κ_1, κ_2 of the proposition are the unique functions consistent with equilibrium. From the above, this transfer function with the conjectured reports leads to a linear quadratic equilibrium value function $V(z, Z)$. Thus, maximizing $V(z + y, Z)$ with respect to y is equivalent to maximizing

$$
\alpha_1(z^i + y) + \alpha_3(z^i + y)^2 + \alpha_5\bar{Z}(z^i + y),
$$

which in turn is equivalent to maximizing

$$
(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i)y + \alpha_3 y^2.
$$

Then, when trader i chooses a report $\tilde{z}$, it must be that this maximizes

$$
(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i)Y^i((\tilde{z}, \hat{z}^{-i})) + \alpha_3 Y^i((\tilde{z}, \hat{z}^{-i}))^2 + T_\kappa^i((\tilde{z}, \hat{z}^{-i}), Z).
$$

Taking a first order condition,

$$
-\frac{n-1}{n}(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i) - \frac{2(n-1)\alpha_3}{n} Y^i((\tilde{z}, \hat{z}^{-i})) + \kappa_1(Z) + 2\kappa_0\left(n\kappa_2(Z) + \tilde{z} + \sum_{j\neq i} \hat{z}^j\right) = 0
$$

Plugging in $\hat{z}^j = z_0^j$ and the function Y^i, we have

$$
\begin{aligned}
&-\frac{n-1}{n}(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i) - \frac{2(n-1)\alpha_3}{n}\left(\frac{-(n-1)\tilde{z}}{n} + \frac{Z - z^i}{n}\right) \\
&\quad + \kappa_1(Z) + 2\kappa_0\left(n\kappa_2(Z) + \tilde{z} - z^i + Z\right) = 0
\end{aligned}
$$

The second order condition is satisfied since $\kappa_0, \alpha_3 < 0$. Since κ_2 is affine, write $\kappa_2(Z) = \hat{a} + \hat{b}Z$. The report $\tilde{z} = z^i$ satisfies this first order condition if

$$-\frac{n-1}{n}(\alpha_1 + \alpha_5 \bar{Z}) - \frac{2(n-1)\alpha_3}{n}\bar{Z} + \kappa_1(Z) + 2\kappa_0 \left(n\hat{a} + n\hat{b}Z + Z\right) = 0.$$

With this,

$$(n\hat{a} + n\hat{b}Z + Z) = \frac{-\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})}{2\kappa_0}$$

so

$$\kappa_2(Z) = \hat{a} + \hat{b}Z = -\bar{Z} + \frac{-\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})}{2\kappa_0 n},$$

implying an equilibrium change in utility of

$$\frac{(-\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}))^2}{4\kappa_0} + \kappa_1(Z)\left(-\bar{Z} + \frac{-\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})}{2\kappa_0 n}\right) + \frac{\kappa_1^2(Z)}{4n^2\kappa_0}$$
$$+ (\kappa_1(Z) - \alpha_1 - \alpha_5\bar{Z})z^i + (\alpha_1 + \alpha_5\bar{Z})\bar{Z} - \alpha_3(z^i)^2 + \alpha_3\bar{Z}^2.$$

This change in utility must be weakly positive for any z and Z. If all traders have $z = \bar{Z}$, then we need that

$$\frac{(-\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}))^2}{4\kappa_0} + \kappa_1(Z)\left(\frac{-\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})}{2\kappa_0 n}\right) + \frac{\kappa_1^2(Z)}{4n^2\kappa_0}$$
$$= -\left(\frac{(-\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}))}{2\sqrt{-\kappa_0}} + \frac{\kappa_1(Z)}{2n\sqrt{-\kappa_0}}\right)^2 \geq 0,$$

which implies that $\kappa_1(Z) = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}$. Plugging this in, we see that

$$\begin{aligned}\hat{a} + \hat{b}Z + z^i &= z^i - \bar{Z} + \frac{-\kappa_1(Z) + \frac{n-1}{n}(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})}{2\kappa_0 n} \\ &= z^i - \bar{Z} - \frac{\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}}{2\kappa_0 n^2}.\end{aligned}$$

So, we see that $n\kappa_2(Z) + \sum_j \hat{z}^j = -(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})/(2\kappa_0 n)$, and thus the equilibrium transfer to trader i is

$$\frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})^2}{4n^2\kappa_0} + (\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z} - \frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})}{2\kappa_0 n^2}\right) + \frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})^2}{4n^2\kappa_0}$$
$$= (\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right) + \frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})^2}{4n^2\kappa_0} - \frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})^2}{2\kappa_0 n^2} + \frac{(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})^2}{4n^2\kappa_0}$$
$$= (\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right).$$

It follows that the equilibrium change in utility for trader i from the mechanism is

$$\begin{aligned}
&(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right) + (\alpha_1 + \alpha_5\bar{Z})(\bar{Z} - z^i) + \alpha_3(\bar{Z})^2 - \alpha_3(z^i)^2 \\
&= 2\alpha_3\bar{Z}z^i - \alpha_3(\bar{Z})^2 - \alpha_3(z^i)^2 \\
&= -\alpha_3(z^i - \bar{Z})^2 \geq 0,
\end{aligned}$$

where the final inequality relies on the fact that α_3 is negative in an equilibrium, from the previous section. Putting this together, as long as $\kappa_1(Z) = \alpha_1 + (\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\bar{Z}$ and $\kappa_2(Z) = \hat{a} + \hat{b}Z$ are given as above, then in equilibrium all traders will find the mechanism ex-post individually rational each time it is run, and their strategy $\hat{z}^i = z^i$ is ex-post optimal. This is true only if $\kappa_1(Z)$ and $\kappa_2(Z)$ take this form.

Finally, since the equilibrium transfers are $(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right)$, we see that the coefficients R_m in

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

are given by

$$\begin{aligned}
R_0 &= 0 \\
R_1 &= -\frac{\alpha_1}{n} \\
R_2 &= -\frac{\alpha_5 + 2\alpha_3}{n^2} \\
R_3 &= \frac{\alpha_5 + 2\alpha_3}{n} \\
R_4 &= \alpha_1.
\end{aligned}$$

Recall from the previous section that

$$\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r + \lambda - 2c} \\
\alpha_5 &= \frac{1}{r + \lambda - c}\left(\frac{c^2}{b} - 2\alpha_3 c + \lambda n R_3\right) \\
\alpha_1 &= \frac{1}{r + \lambda - c}\left(rv + \frac{ac}{b} + \lambda R_4\right),
\end{aligned}$$

so, plugging in R_3, R_4, and rearranging,

$$\alpha_3 = \frac{-\gamma}{r+\lambda-2c}$$
$$\alpha_5 = \frac{1}{r-c}(\frac{c^2}{b} - 2\alpha_3 c + 2\lambda\alpha_3)$$
$$\alpha_1 = \frac{1}{r-c}(rv + \frac{ac}{b}).$$

C.4 Solving the HJB Equation

From the above, the value function takes the form

$$V(z^i, Z) = \alpha_0^i + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3 (z^i)^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z^i \bar{Z}.$$

The associated HJB equation is

$$0 = -\gamma(z^i)^2 + r(vz^i - V(z^i, Z)) + \frac{\sigma_i^2}{2} V_{zz}(z, Z) + \frac{\sigma_Z^2}{n^2} V_{ZZ}(z^i, Z) + 2\frac{\rho^i}{n} V_{zZ}(z^i, Z)$$
$$+ \sup_{D, \hat{z}^i} -\Phi_{(a,b,c)}(D; Z - z^i)D + V_z(z^i, Z)D + \lambda \left(V(z^i + Y^i((\hat{z}^i, \hat{z}^{-i})), Z) - V(z, Z) + T_\kappa^i((\hat{z}^i, \hat{z}^{-i}), Z) \right).$$

From the previous subsection, we know that fixing the equilibrium reports $\hat{z}^{-i}$ of the other traders, the report $\hat{z}^i = z^i$ achieves the supremum in the HJB equation for any D, as long as

$$\kappa_2(Z) = \hat{a} + \hat{b}Z = -\bar{Z} - \frac{\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}}{2\kappa_0 n^2}.$$

Since $V_z = \alpha_1 + 2\alpha_3 z^i + \alpha_5 \bar{Z}$, following steps that are identical to those of the proof of Proposition 3, and as long as $b < 0$, the unique demand that achieves the maximum in the HJB equation is

$$D = -\frac{1}{2}[(n-1)a + n(-b\phi - a) - cz^i + b(n-1)\left(\alpha_1 + 2\alpha_3 z^i + \alpha_5 \bar{Z}\right)].$$

Plugging in $Z = n(-b\phi - a)/c$,

$$D = -\frac{1}{2}[(n-1)a + n(-b\phi - a) - cz^i + b(n-1)\left(\alpha_1 + 2\alpha_3 z^i + \alpha_5 \frac{-b\phi - a}{c}\right)].$$

Recall from the previous section that, after plugging in equilibrium transfers,

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda-2c} \\
\alpha_5 &= \frac{1}{r-c}(\frac{c^2}{b} - 2\alpha_3 c + 2\lambda\alpha_3) \\
\alpha_1 &= \frac{1}{r-c}(rv + \frac{ac}{b}).
\end{aligned}
$$

Then, matching coefficients in the expression for D, we have

$$
\begin{aligned}
c &= -\frac{1}{2}[-c + 2b(n-1)\alpha_3] \\
b &= -\frac{1}{2}[-nb + b(n-1)\left(\frac{1}{r-c}[2\alpha_3 b - c - \lambda 2\alpha_3 \frac{b}{c}]\right) \\
a &= -\frac{1}{2}[-a + b(n-1)\frac{1}{r-c}\left(rv + 2\lambda\alpha_3(\frac{-a}{c}) + 2\alpha_3 a\right)].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
c &= -2b(n-1)\alpha_3 \\
(r-c)(n-2) &= \left[2\alpha_3 b(n-1) - c(n-1) - \lambda 2\alpha_3 \frac{b}{c}(n-1)\right] \\
r(n-2) &= -2c + \lambda \\
c &= \frac{\lambda - r(n-2)}{2} \\
\alpha_3 &= \frac{-\gamma}{r(n-1)} \\
b &= \frac{r\lambda - r^2(n-2)}{4\gamma}.
\end{aligned}
$$

From this, we see that b is strictly negative, satisfying the second order condition, if and only if $\lambda < r(n-2)$.

Next, we have

$$
\begin{aligned}
a &= \frac{1}{r-c}\left(-b(n-1)rv + 2\lambda\alpha_3 b(n-1)\frac{a}{c} - 2\alpha_3 ab(n-1)\right) \\
&= \frac{1}{r-c}\left(-b(n-1)rv + -\lambda a + ca\right) \\
&= \frac{2}{rn-\lambda}\left(-\frac{r\lambda - r^2(n-2)}{4\gamma}(n-1)rv + a\frac{-\lambda - r(n-2)}{2}\right).
\end{aligned}
$$

Noting that

$$\frac{\lambda + r(n-2)}{rn - \lambda} + 1 = \frac{2r(n-1)}{rn-\lambda},$$

we see that

$$a = -\frac{(r\lambda - r^2(n-2))\, v}{4\gamma}.$$

From this, we see that $a = -vb$ and $c = 2\gamma b/r$ so

$$\phi_t = \frac{a + c\bar{Z}_t}{-b} = v - \frac{2\gamma}{r}\bar{Z}_t$$

and

$$\begin{aligned}\alpha_1 &= \frac{1}{r-c}(rv + \frac{ac}{b}) \\ &= \frac{1}{r-c}\,(rv - vc) = v.\end{aligned}$$

Likewise,

$$\begin{aligned}\alpha_5 + 2\alpha_3 &= \frac{1}{r-c}(\frac{c^2}{b} - 2\alpha_3 c + 2\lambda\alpha_3) + 2\alpha_3 \\ &= \frac{1}{r-c}(\frac{2\gamma}{r}c + 2\alpha_3(r-c) - 2\alpha_3 c + 2\lambda\alpha_3) \\ &= \frac{1}{r-c}(\frac{2\gamma}{r}c + 2\alpha_3(r + \lambda - 2c)) \\ &= \frac{1}{r-c}(\frac{2\gamma}{r}c - 2\gamma) = \frac{-2\gamma}{r}.\end{aligned}$$

It follows that

$$\alpha_5 = \frac{-2\gamma}{r} - 2\alpha_3 = \frac{-2\gamma}{r} + \frac{2\gamma}{r(n-1)}.$$

Plugging in $\alpha_1, \alpha_5, \alpha_3$ into the equilibrium $\kappa_2(Z)$, we see that

$$\begin{aligned}\kappa_2(Z) &= -\bar{Z} - \frac{\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}}{2\kappa_0 n^2} \\ \kappa_2(Z) &= -\bar{Z} - \frac{v - \frac{2\gamma}{r}\bar{Z}}{2\kappa_0 n^2},\end{aligned}$$

and likewise

$$\kappa_1(Z) = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z} = v - \frac{2\gamma}{r}\bar{Z}.$$

Recalling that $R_1 = -\alpha_1/n$ and $\alpha_1 = v$, the formula for α_2 is

$$\begin{aligned}\alpha_2 &= \frac{1}{r}(\frac{ca}{-b} + (\lambda - c)\alpha_1 + \lambda n R_1) \\ &= \frac{1}{r}(cv + (\lambda - c)v - \lambda\alpha_1) = 0.\end{aligned}$$

Recalling that

$$R_2 = -\frac{\alpha_5 + 2\alpha_3}{n^2} = \frac{2\gamma}{rn^2},$$

the formula for α_4 is

$$\begin{aligned}\alpha_4 &= \frac{1}{r}(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2) \\ &= \frac{1}{r}(\frac{-2\gamma}{r}c + (\lambda - c)\alpha_5 + \lambda\alpha_3 - \lambda(\alpha_5 + 2\alpha_3)) \\ &= \frac{1}{r}(\frac{-2\gamma}{r}c - c(\alpha_5 + 2\alpha_3) + (2c - \lambda)\alpha_3) \\ &= \frac{1}{r}((2c - \lambda - r)\alpha_3 + r\alpha_3) \\ &= \frac{1}{r}(\gamma - \frac{\gamma}{(n-1)}) = \frac{\gamma(n-2)}{r(n-1)}.\end{aligned}$$

Finally, since $R_0 = 0$, the formula for α_0^i is

$$\begin{aligned}\alpha_0^i &= \frac{1}{r}(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \lambda R_0) \\ &= \frac{1}{r}(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}),\end{aligned}$$

and since $\alpha_1 - \alpha_5$ are exactly the same as in proposition 3, $\alpha_0^i = \theta_i$ from the statement of proposition 3. It follows the value function is the same as that of proposition 3.

C.5 Completing the Verification

We have shown that in a symmetric equilibrium, value functions are linear-quadratic and in particular must be twice continuously differentiable. The HJB of the previous subsection is thus a necessary condition, and there is a unique candidate linear-quadratic equilibrium which satisfies it. We have shown that if each player follows their linear strategy, they indeed get their candidate value function as a continuation value. It remains to show that each player prefers this to any other strategy.

We take the notation of Section C.2. Fix the $a, b, c, \kappa_0, \kappa_1(Z), \kappa_2(Z)$ of the previous subsection, and the corresponding constants $\alpha_0^i, \alpha_1 - \alpha_5$ for some player i. We fix some admissible demand process D^i, and report process $\tilde{z}$, by which the inventory of trader i at time t is

$$z_t^{(D,\tilde{z})} = z_0^i + \int_0^t D_s^i ds + H_t^i + \int_0^t Y^i((\tilde{z}_s, \hat{z}_s^{-i}))dN_s. \tag{81}$$

Following the steps of the derivation of the value function, we can show that under the laws of motion implied by $D^i, \tilde{z}$,

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}[\int_{0+}^{\mathcal{T}} D_s^i(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z_{s-}^{D,\tilde{z}}) + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_3 \sigma_i^2 + \alpha_5 \frac{\rho^i}{n}$$
$$+ \lambda Y^i((\tilde{z}_s, \hat{z}_s^{-i}))(\alpha_1 + \alpha_5 \bar{Z}_{s-} + 2\alpha_3 z_{s-}^{D,\tilde{z}} + \alpha_3 Y^i((\tilde{z}_s, \hat{z}_s^{-i}))) + r(v z_s^{D,\tilde{z}} - V(z_s^{D,\tilde{z}}, Z_s))ds].$$

Since $\alpha_0 - \alpha_5$ satisfy the HJB, and using the fact that

$$\mathbb{E}\left[\int_0^{\mathcal{T}} \lambda T_\kappa^i((\tilde{z}_s, \hat{z}_s^{-i}), Z_s)\, ds\right] = \mathbb{E}\left[\int_0^{\mathcal{T}} T_\kappa^i((\tilde{z}_s, \hat{z}_s^{-i}), Z_s)\, dN_s\right],$$

we have

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] \leq \mathbb{E}[\int_{0+}^{\mathcal{T}} D_s^i \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^{D,\tilde{z}}) + \gamma(z_s^{D,\tilde{z}})^2 ds$$
$$- \int_0^{\mathcal{T}} T_\kappa^i((\hat{z}_s^i, \hat{z}_s^{-i}), Z_s)dN_s].$$

Rearranging, this is

$$V(z_0^i, Z_0) \geq \mathbb{E}[\pi z_{\mathcal{T}}^{D,\tilde{z}} + \int_{0+}^{\mathcal{T}} -D_s^i \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^{D,\tilde{z}}) - \gamma(z_s^{D,\tilde{z}})^2 ds$$
$$+ \int_0^{\mathcal{T}} T_\kappa^i((\hat{z}_s^i, \hat{z}_s^{-i}), Z_s)dN_s].$$

Since this holds with equality for the conjectured linear strategy, the linear strategy is optimal.

D Proof of Proposition 5

The proof proceeds in 6 steps. First, we show that transfers take a particular quadratic form in any equilibrium. Second, we show that $r + \lambda - 2c > 0$ in any equilibrium. (If not, some trader is using an inadmissible or suboptimal strategy.) Third, we show that, given the quadratic form of the transfer function, the value function in any equilibrium must take a particular linear-quadratic form. Fourth, we characterize the optimal mechanism reports and corresponding equilibrium transfers, and characterize equilibrium individual rationality (IR). Fifth, we explicitly solve for the coefficients of the value function and for the strategies that attain the maxima in the HJB equation. Finally, we show that for these candidate optimal strategies, every trader receives an inferior payoff if using any alternative strategy.

D.1 Equilibrium Transfers

We fix a symmetric equilibrium $\mathcal{C} = (a, b, c)$. First, we recall that in a symmetric linear-quadratic equilibrium, the market clearing price process ϕ must satisfy

$$na + nb\phi_t + cZ_t = 0,$$

which implies that

$$\phi_t = \frac{a + c\bar{Z}_t}{-b},$$

and $a + b\phi_t + cz_t^i = c(z_t^i - \bar{Z}_t)$.

Recall that the transfers are given by

$$\hat{T}^i(\hat{z}; p) = \kappa_0 \left(-n\delta(p) + \sum_{j=1}^{n} \hat{z}^j \right)^2 + p\left(\hat{z}^i - \delta(p)\right) + \frac{p^2}{4\kappa_0 n^2}, \tag{82}$$

where δ is an affine function. In equilibrium, ϕ_t is affine in Z_t, and everyone reports $\hat{z}^j = z^j$. It is straightforward to show then that in any symmetric equilibrium, the transfers are of the form

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i$$

for constants R_0 through R_4 that depend on δ, κ_0, and the equilibrium coefficients (a, b, c).

D.2 Admissibility

Fix a symmetric equilibrium $\mathcal{C} = (a, b, c)$. The inventory of trader i is

$$z_t^i = z_0^i + c \int_0^t z_s^i - \bar{Z}_s \, ds + H_t^i - \int_0^t (z_{s-}^i - \bar{Z}_{s-}) \, dN_s. \tag{83}$$

Since, for fixed c, this is identical to the same inventory evolution in proposition 4 (section C.1), the exact same proof can be used to show that

$$\mathbb{E}\left[\int_0^{\mathcal{T}} (z_s^i)^2 \, ds\right]$$

is finite if and only if $2c < r + \lambda$.

D.3 The value function

We claim that in any linear-quadratic symmetric equilibrium, the value function

$$V(z,Z)=\mathbb{E}\left[\pi z^i_{\mathcal{T}}+\int_0^{\mathcal{T}}(-\gamma(z^i_s)^2-c(z^i_s-\bar{Z}_s)\left(\frac{a+c\bar{Z}_s}{-b}\right)\,ds)+\int_0^{\mathcal{T}}\hat{T}^i(\hat{z}_s;\phi_{s-})\,dN_s\right]$$

takes the form

$$V(z,Z)=\alpha_0^i+\alpha_1 z+\alpha_2\bar{Z}+\alpha_3 z^2+\alpha_4\bar{Z}^2+\alpha_5 z\bar{Z},$$

where

$$\begin{aligned}
\alpha_3&=\frac{-\gamma}{r+\lambda-2c}\\
\alpha_5&=\frac{1}{r+\lambda-c}(\frac{c^2}{b}-2\alpha_3 c+\lambda nR_3)\\
\alpha_4&=\frac{1}{r}(\frac{c^2}{-b}+(\lambda-c)\alpha_5+\lambda\alpha_3+\lambda n^2R_2)\\
\alpha_1&=\frac{1}{r+\lambda-c}(rv+\frac{ac}{b}+\lambda R_4)\\
\alpha_2&=\frac{1}{r}(\frac{ca}{-b}+(\lambda-c)\alpha_1+\lambda nR_1)\\
\alpha_0^i&=\frac{1}{r}(\alpha_3\sigma_i^2+\alpha_4\frac{\sigma_Z^2}{n^2}+\alpha_5\frac{\rho^i}{n}+\lambda R_0).
\end{aligned}$$

where R_0 through R_4 are the previously defined transfer coefficients. Given the α coefficients, we have

$$\begin{aligned}
&(r+\lambda)\left(\alpha_0^i+\alpha_1 z+\alpha_2\bar{Z}+\alpha_3 z^2+\alpha_4\bar{Z}^2+\alpha_5 z\bar{Z}\right)\\
&\quad=rvz-\gamma z^2+\alpha_4\frac{\sigma_Z^2}{n^2}+\alpha_3\sigma_i^2+\alpha_5\frac{\rho^i}{n}-c(z-\bar{Z})\frac{a+c\bar{Z}}{-b}\\
&\qquad+c(z-\bar{Z})(\alpha_1+2\alpha_3 z+\alpha_5\bar{Z})+\lambda(\alpha_0^i+\alpha_1\bar{Z}+\alpha_2\bar{Z}+\alpha_3\bar{Z}^2\\
&\qquad+\alpha_4\bar{Z}^2+\alpha_5\bar{Z}^2+R_0+R_1Z+R_2Z^2+R_3Zz+R_4z).
\end{aligned}$$

The rest of the proof proceeds exactly as in section C.2, and is thus omitted.

D.4 Optimal Mechanism Reports and Equilibrium IR

In the HJB equation, trader i chooses a demand D and a report $\hat{z}^i$ to maximize[30]

$$\sup_{D,\hat{z}^i}-D\Phi_{(a,b,c)}(D;Z-z^i)+DV_z(z^i,Z)+\lambda(V(z^i+Y^i((\hat{z}^i,\hat{z}^{-i})),Z)+\hat{T}^i((\hat{z}^i,\hat{z}^{-i});\Phi_{(a,b,c)}(D;Z-z^i))).$$

[30] For the purpose of this proof, we suppose trader i can observe Z_t. We show the corresponding optimal strategy depends only on the information in information set of trader i (which does not include Z_t). Because the resulting strategy is optimal even in the larger set of strategies, it is optimal with respect to strategies that are adapted to the information filtration of trader i.

In any linear symmetric equilibrium, trader i must have a value function of the specified form. Thus, maximizing $V(z^i+y,Z)$ is equivalent to maximizing

$$\alpha_1(z^i+y)+\alpha_3(z^i+y)^2+\alpha_5\bar{Z}(z^i+y),$$

which is equivalent to maximizing

$$(\alpha_1+\alpha_5\bar{Z})y+\alpha_3y^2+2\alpha_3z^iy.$$

If trader i chooses the auction demand D, thus setting the price $\phi=\Phi_{(a,b,c)}(D;Z-z^i)$ that would be used in the mechanism if one were held immediately, and given that the total of the other traders' reports is $\sum_{j\neq i}z^j=Z-z^i$, trader i gets a transfer of

$$\kappa_0\left(-n\delta(p)+Z-z^i+\hat{z}^i\right)^2+p\left(\hat{z}^i-\delta(p)\right)+\frac{p^2}{4\kappa_0n^2}, \tag{84}$$

and a reallocation of

$$Y^i((\hat{z}^i,\hat{z}^{-i}))=\frac{Z-z^i}{n}-\frac{n-1}{n}\hat{z}^i.$$

Thus, the optimization problem faced by trader i is equivalent to maximizing the sum of (i) the quantity $-D\Phi_{(a,b,c)}(D;Z-z^i)+DV_z(z^i,Z)$ and (ii) the product of λ with

$$\begin{aligned}\mathcal{E}(\phi,Z,z^i,\hat{z}^i)\equiv{}&(\alpha_1+\alpha_5\bar{Z})(\frac{Z-z^i}{n}-\frac{n-1}{n}\hat{z}^i)+\alpha_3(\frac{Z-z^i}{n}-\frac{n-1}{n}\hat{z}^i)^2\\&+2\alpha_3z^i(\frac{Z-z^i}{n}-\frac{n-1}{n}\hat{z}^i)+\kappa_0(-n\delta(\phi)+Z-z^i+\hat{z}^i)^2+\phi(\hat{z}^i-\delta(\phi))+\frac{\phi^2}{4\kappa_0n^2},\end{aligned}$$

evaluated at $\phi=\Phi_{(a,b,c)}(D;Z-z^i)$.

The first order condition for optimality of $\hat{z}^i$ is

$$\begin{aligned}\frac{\partial\mathcal{E}(\phi,Z,z^i,\hat{z}^i)}{\partial\hat{z}^i}={}&-\frac{n-1}{n}(\alpha_1+\alpha_5\bar{Z})+\frac{2(n-1)^2}{n^2}\alpha_3\hat{z}^i-2\frac{n-1}{n}\alpha_3\frac{Z-z^i}{n}\\&-\frac{n-1}{n}2\alpha_3z^i+2\kappa_0(-n\delta(\phi)+\hat{z}^i+Z-z^i)+\phi=0.\end{aligned}$$

The second-order condition is satisfied if $\alpha_3<0$ and $\kappa_0<0$. For the candidate equilibrium strategy $\hat{z}^i=z^i$, we have

$$\frac{\partial\mathcal{E}(\phi,Z,z^i,\hat{z}^i)}{\partial\hat{z}^i}=-\frac{n-1}{n}(\alpha_1+\alpha_5\bar{Z})+\frac{2(n-1)\alpha_3}{n}(-\bar{Z})+2\kappa_0(-n\delta(\phi)+Z)+\phi.$$

Plugging in

$$Z=n\frac{-b\phi-a}{c},$$

which must hold in a symmetric equilibrium, and writing $\delta(\phi) = -\hat{a} - \hat{b}\phi$, we have

$$\frac{\partial \mathcal{E}(\phi, Z, z^i, \hat{z}^i)}{\partial \hat{z}^i} = -\frac{n-1}{n}\left(\alpha_1 + \alpha_5 \frac{-b\phi - a}{c}\right) + \frac{2(n-1)\alpha_3}{n}\frac{b\phi + a}{c}$$
$$+ 2\kappa_0\left(n\hat{a} + n\hat{b}\phi + n\frac{-b\phi - a}{c}\right) + \phi.$$

The candidate equilibrium strategy $\hat{z}^i$ is therefore optimal provided that

$$0 = -\frac{n-1}{n}(\alpha_1 - \frac{\alpha_5 a}{c}) + \frac{2(n-1)\alpha_3 a}{nc} + 2\kappa_0 n\hat{a} - \frac{2na\kappa_0}{c}$$
$$0 = \frac{n-1}{n}(\frac{\alpha_5 b}{c}) + \frac{2(n-1)\alpha_3 b}{nc} + 2\kappa_0 n(\hat{b} - \frac{b}{c}) + 1,$$

or equivalently,

$$\hat{a} = \frac{a}{c} - \frac{1}{2n\kappa_0}(-\frac{n-1}{n}(\alpha_1 - \frac{\alpha_5 a}{c}) + \frac{2(n-1)\alpha_3 a}{nc})$$
$$\hat{b} = \frac{b}{c} - \frac{1}{2n\kappa_0}(\frac{n-1}{n}(\frac{\alpha_5 b}{c}) + \frac{2(n-1)\alpha_3 b}{nc} + 1).$$

These equations imply that

$$\begin{aligned}\nu \equiv\ & n\hat{a} + n\hat{b}\frac{a + c\bar{Z}}{-b} + Z \\ =& -\frac{1}{2\kappa_0}\left(-\frac{n-1}{n}(\alpha_1 - \frac{\alpha_5 a}{c}) + \frac{2(n-1)\alpha_3 a}{nc}\right) \\ &- \frac{1}{2\kappa_0}\left(\frac{a + c\bar{Z}}{-b}\right)\left(\frac{n-1}{n}(\frac{\alpha_5 b}{c}) + \frac{2(n-1)\alpha_3 b}{nc} + 1\right).\end{aligned}$$

Evaluating this expression for ν at $\phi = -(a + c\bar{Z})/b$, we have

$$\nu = \frac{-1}{2\kappa_0}\left(\phi - \frac{n-1}{n}\alpha_1 + \frac{n-1}{n}\alpha_5 \frac{a + b\phi}{c} + \frac{2(n-1)\alpha_3}{n}\frac{a + b\phi}{c}\right). \tag{85}$$

Consider the ex-post equilibrium IR condition that the transfer plus $V(\bar{Z}, Z) - V(z^i, Z)$ must be weakly positive. This must hold even when all traders have inventory $\bar{Z}$ going into the mechanism. In particular, the sum of the transfers must be weakly positive in this case, but it is always weakly negative by budget balance, so the transfers must sum to 0. In general, the sum of the transfers is

$$-n(\sqrt{-\kappa_0}\ (-n\delta(\phi) + \sum_j \hat{z}^j) - \frac{\phi}{2n\sqrt{-\kappa_0}})^2.$$

So, if the transfers are to sum to 0, it must be that

$$\sqrt{-\kappa_0}\ (-n\delta(\phi)+\sum_j \hat{z}^j)-\frac{\phi}{2n\sqrt{-\kappa_0}}=0$$

$$|\kappa_0|(-n\delta(\phi)+\sum_j \hat{z}^j)-\frac{\phi}{2n}=-\kappa_0(-n\delta(\phi)+\sum_j \hat{z}^j)-\frac{\phi}{2n}=0. \tag{86}$$

Recall from equation (85) that at the equilibrium strategies and the $\delta(\phi)$ consistent with IC,

$$-n\delta(\phi)+\sum_j \hat{z}^j=\frac{-1}{2\kappa_0}\left(\phi-\frac{n-1}{n}\alpha_1+\frac{n-1}{n}\alpha_5\frac{a+b\phi}{c}+\frac{2(n-1)\alpha_3}{n}\frac{a+b\phi}{c}\right).$$

Thus for IR to hold, combining this with equation (86), it must be that

$$\begin{aligned}
&\frac{1}{2}\left(\frac{n-1}{n}\phi-\frac{n-1}{n}\alpha_1+\frac{n-1}{n}\alpha_5\frac{a+b\phi}{c}+\frac{2(n-1)\alpha_3}{n}\frac{a+b\phi}{c}\right)\\
&\quad=\frac{1}{2}\left((\frac{n-1}{n})\phi-\frac{n-1}{n}\alpha_1-\frac{n-1}{n}\alpha_5\bar{Z}-\frac{2(n-1)\alpha_3}{n}\bar{Z}\right)\\
&\quad=0.
\end{aligned}$$

Put differently, for the equilibrium strategies to be IR, we need the condition

$$\phi=\alpha_1+(\alpha_5+2\alpha_3)\bar{Z}. \tag{87}$$

We conjecture and later verify that (87) holds in equilibrium. Given this, we see that, in equilibrium,

$$-n\delta(\phi)+\sum_j \hat{z}^j=\frac{-\phi}{2\kappa_0 n}.$$

Likewise, we see that

$$\begin{aligned}
&-\delta(\phi)+\hat{z}^i=\hat{a}+\hat{b}\frac{a+c\bar{Z}}{-b}+z^i\\
&\quad=z^i-\bar{Z}-\frac{1}{2\kappa_0 n}\left(\phi-\frac{n-1}{n}\alpha_1+\frac{n-1}{n}\alpha_5\frac{a+b\phi}{c}+\frac{2(n-1)\alpha_3}{n}\frac{a+b\phi}{c}\right)\\
&\quad=z^i-\bar{Z}-\frac{\phi}{2\kappa_0 n^2}.
\end{aligned}$$

Now, if we plug $\delta(\phi)=-\hat{a}-\hat{b}\phi$ into the definition of $\mathcal{E}(\phi,Z,z^i,\hat{z}^i)$, we arrive at

$$
\begin{aligned}
\mathcal{E}(\phi, Z, z^i, \hat{z}^i) &= (\alpha_1 + \alpha_5 \bar{Z})(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i) + \alpha_3(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i)^2 \\
&\quad + 2\alpha_3 z^i(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i) + \kappa_0(n(\hat{a} + \hat{b}\phi) + Z - z^i + \hat{z}^i)^2 + \phi(\hat{z}^i + (\hat{a} + \hat{b}\phi)) + \frac{\phi^2}{4\kappa_0 n^2},
\end{aligned}
$$

The partial derivative of $\mathcal{E}(\phi, Z, z^i, \hat{z}^i)$ with respect to ϕ is then

$$
\mathcal{E}_\phi(\phi, Z, z^i, \hat{z}^i) = 2\kappa_0 n\hat{b}(n(\hat{a} + \hat{b}\phi) + Z - z^i + \hat{z}^i) + (\hat{z}^i + (\hat{a} + 2\hat{b}\phi)) + \frac{\phi}{2\kappa_0 n^2}.
$$

Plugging in the candidate $\hat{z}^i = z^i$ and the fact from above that $\hat{a} + \hat{b}\phi = -\bar{Z} - \phi/(2\kappa_0 n^2)$,

$$
\mathcal{E}_\phi(\phi, Z, z^i, \hat{z}^i) = 2\kappa_0 n\hat{b}\frac{-\phi}{2\kappa_0 n} + \hat{b}\phi + (z^i - \bar{Z} - \frac{\phi}{2\kappa_0 n^2}) + \frac{\phi}{2\kappa_0 n^2} = z^i - \bar{Z}.
$$

Finally, using the equilibrium reports and the δ consistent with IC, equilibrium transfers are

$$
\begin{aligned}
\kappa_0\left(-n\delta(\phi) + \sum_j \hat{z}^j\right)^2 + \phi(\hat{z}^i - \delta(\phi)) + \frac{\phi^2}{4\kappa_0 n^2} &= \frac{\phi^2}{4\kappa_0 n^2} + \phi\left(z^i - \bar{Z} - \frac{\phi}{2\kappa_0 n^2}\right) + \frac{\phi^2}{4\kappa_0 n^2} \\
&= \phi(z^i - \bar{Z}) \\
&= \frac{a + c\bar{Z}}{-b}(z^i - \bar{Z}),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
R_0 &= 0 \\
R_1 &= \frac{a}{nb} \\
R_2 &= \frac{c}{n^2 b} \\
R_3 &= \frac{c}{-nb} \\
R_4 &= \frac{a}{-b}.
\end{aligned}
$$

D.5 Solving the HJB

The optimization solved is

$$
\sup_{D, \hat{z}^i} -D\Phi_{(a,b,c)}(D; Z - z^i) + DV_z(z^i, Z) + \lambda\mathcal{E}(\Phi_{(a,b,c)}(D; Z - z^i), Z, z^i, \hat{z}^i)
$$

Taking a total derivative with respect to $D, \hat{z}^i$, we need

$$-\Phi_{(a,b,c)}(D;Z-z^i)-D\Phi'_{(a,b,c)}(D;Z-z^i)+V_z(z^i,Z)+\lambda\Phi'_{(a,b,c)}(D;Z-z^i)\mathcal{E}_\phi(\Phi_{(a,b,c)}(D;Z-z^i),Z,z^i,\hat{z}^i)=0$$

$$\mathcal{E}_{\hat{z}^i}(\Phi_{(a,b,c)}(D;Z-z^i),Z,z^i,\hat{z}^i)=0,$$

and both of these must hold with $D=a+b\phi+cz^i$ (implying $\Phi_{(a,b,c)}(D;Z-z^i)=\frac{a+c\bar{Z}}{-b}$) and $\hat{z}^i=z^i$. Recall $\Phi'_{(a,b,c)}(D;Z-z^i)=\frac{-1}{b(n-1)}$. From the above, the second equation is satisfied at $\phi=\frac{a+c\bar{Z}}{-b}$ and the conjectured $\hat{z}^i$ as long as

$$0=-\frac{n-1}{n}(\alpha_1-\frac{\alpha_5 a}{c})+\frac{2(n-1)\alpha_3 a}{nc}+2\kappa_0 n\hat{a}-\frac{2na\kappa_0}{c} \tag{88}$$
$$0=\frac{n-1}{n}(\frac{\alpha_5 b}{c})+\frac{2(n-1)\alpha_3 b}{nc}+2\kappa_0 n(\hat{b}-\frac{b}{c})+1, \tag{89}$$

where we've written $\delta(\phi)$ as $\delta(\phi)=-\hat{a}-\hat{b}\phi$. For the FOC on D, we need

$$-\phi+\frac{1}{b(n-1)}(a+b\phi+cz^i)+(\alpha_1+2\alpha_3 z^i+\alpha_5\bar{Z})-\frac{\lambda}{b(n-1)}\mathcal{E}_\phi(\phi,Z,z^i,\hat{z}^i)=0.$$

We showed that at equilibrium $\mathcal{E}_\phi=z^i-\bar{Z}$. Plug in this and $\bar{Z}=\frac{-b\phi-a}{c}$, to see that

$$-\phi+\frac{1}{b(n-1)}(a+b\phi+cz^i)+(\alpha_1+2\alpha_3 z^i+\alpha_5\frac{-b\phi-a}{c})-\frac{\lambda}{b(n-1)}(z^i-\frac{-b\phi-a}{c})=0,$$

or, gathering terms,

$$\begin{aligned}
0&=-1+\frac{1}{(n-1)}-\alpha_5\frac{b}{c}-\frac{\lambda}{c(n-1)}\\
0&=\frac{1}{b(n-1)}c+2\alpha_3-\frac{\lambda}{b(n-1)}\\
0&=\frac{1}{b(n-1)}a+(\alpha_1+\alpha_5\frac{-a}{c})-\frac{\lambda}{b(n-1)}\frac{a}{c}.
\end{aligned}$$

Rearranging,

$$0=-(n-2)c-\alpha_5(n-1)b-\lambda \tag{90}$$
$$c=-2\alpha_3 b(n-1)+\lambda, \tag{91}$$

while from the derivation of the linear quadratic value function,

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda-2c} \\
\alpha_5 &= \frac{1}{r+\lambda-c}(\frac{c^2}{b} - 2\alpha_3 c + n\lambda R_3),
\end{aligned}
$$

where R_3 is the coefficient on Zz in the transfer. From the last section, in equilibrium we have $R_3 = c/(-nb)$ and thus the relevant system is

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda-2c} \\
\alpha_5 &= \frac{1}{r+\lambda-c}(\frac{c^2}{b} - 2\alpha_3 c - \frac{\lambda c}{b}).
\end{aligned}
$$

Multiplying both sides of the α_5 equation by $b(n-1)$, we have

$$
\alpha_5 b(n-1) = \frac{1}{r+\lambda-c}(c^2(n-1) - 2\alpha_3 b(n-1)c - \lambda c(n-1)),
$$

and plugging in the above,

$$
\alpha_5 b(n-1) = \frac{nc}{r+\lambda-c}(c-\lambda),
$$

so

$$
\begin{aligned}
0 &= -(n-2)c - (\frac{nc}{r+\lambda-c}(c-\lambda)) - \lambda \\
0 &= -(n-2)c(r+\lambda-c) - nc(c-\lambda) - \lambda(r+\lambda-c) \\
0 &= -2c^2 + c(-(n-2)(r+\lambda) + n\lambda + \lambda) - \lambda(r+\lambda) \\
0 &= -2c^2 + c(-(n-2)r + 3\lambda) - \lambda(r+\lambda) \\
c &= \frac{(-(n-2)r+3\lambda) \pm \sqrt{(-(n-2)r+3\lambda)^2 - 8\lambda(r+\lambda)}}{4}.
\end{aligned}
$$

It is clear that either both or neither of these roots are real. By the Descartes rule of signs, if both are real, they are either both positive, or neither are positive. In particular, assuming that $(-(n-2)r+3\lambda)^2 - 8\lambda(r+\lambda) > 0$ so that both roots exist, if we can show one is negative then they both are negative. If $-(n-2)r+3\lambda < 0$, then the smaller root must be negative and we are done. If $-(n-2)r+3\lambda \geq 0$, then the larger root is positive so both roots are positive. Thus we see we need that $-(n-2)r+3\lambda < 0$ and $(-(n-2)r+3\lambda)^2 - 8\lambda(r+\lambda) \geq 0$, which can be concisely written as

$$
-(n-2)r + 3\lambda \leq -\sqrt{8\lambda(r+\lambda)}.
$$

Define

$$F(c,\lambda) = -2c^2 + c(-(n-2)r + 3\lambda) - \lambda(r+\lambda),$$

and note from the above that $F(c,\lambda) = 0$ implies an equilibrium, as long as $c < 0$ such that $b < 0$ and the second order condition above holds.

We have that $F_{cc} = -4 < 0$ and $\lim_{c\to-\infty} F = \lim_{c\to\infty} F = -\infty$. Thus, as c increases from negative infinity to infinity, F_c crosses from positive to negative exactly once, at

$$c_0 = \frac{-(n-2)r + 3\lambda}{4}.$$

Since there are two roots, we see the derivative F_c must be positive at the smaller root $\underline{c}(\lambda)$ and negative at the larger root $\overline{c}(\lambda)$, so $\underline{c}(\lambda) < c_0 < \overline{c}(\lambda)$. Fix a $\lambda \in (0, \bar{\lambda})$ and consider small, disjoint neighborhoods around $(\lambda, \overline{c}(\lambda))$ and $(\lambda, \underline{c}(\lambda))$. Applying implicit function theorem to each of these functions,

$$\begin{aligned}
\frac{\partial c}{\partial \lambda} &= -\frac{F_\lambda}{F_c} \\
&= -\frac{-r - 2\lambda + 3c}{F_c}
\end{aligned}$$

Since $c < 0$ in either equilibrium, the numerator is always negative. We just showed F_c is positive at the smaller root and thus $\frac{\partial \underline{c}(\lambda)}{\partial \lambda} > 0$ so that c increases monotonically in λ.

Now, recall we have

$$(r + \lambda - 2c)\alpha_3 = -\gamma,$$

which, combined with equation (91), implies

$$\begin{aligned}
c(r+\lambda-2c) &= -2\alpha_3 b(n-1)(r+\lambda-2c) + \lambda(r+\lambda-2c) \\
c(r+\lambda-2c) &= 2\gamma b(n-1) + \lambda(r+\lambda-2c).
\end{aligned}$$

Using the above quadratic equation for c, this can be rewritten

$$\begin{aligned}
c(r+\lambda) - (c(-(n-2)r+3\lambda) - \lambda(r+\lambda)) &= 2\gamma b(n-1) + \lambda(r+\lambda-2c) \\
c(r+\lambda) - (c(-(n-2)r+3\lambda)) &= 2\gamma b(n-1) - 2\lambda c \\
cr(n-1) &= 2\gamma b(n-1) \\
c &= \frac{2\gamma}{r} b,
\end{aligned}$$

which implies that

$$b = \frac{r^2}{8\gamma}\left(-(n-2) + \frac{3\lambda}{r} \pm \sqrt{(-(n-2) + \frac{3\lambda}{r})^2 - \frac{8\lambda(r+\lambda)}{r^2}}\right).$$

Note

$$[\frac{3\lambda}{r}-(n-2)]^2-\frac{8\lambda(r+\lambda)}{r^2}=\frac{\lambda^2}{r^2}-\frac{6\lambda(n-2)}{r}+(n-2)^2-\frac{8\lambda}{r} \tag{92}$$

$$=(\frac{\lambda}{r}-(n-2))^2-\frac{4\lambda n}{r}, \tag{93}$$

so we have shown that

$$b=\frac{-r^2}{8\gamma}\left((n-2)-\frac{3\lambda}{r}\pm\sqrt{(\frac{\lambda}{r}-(n-2))^2-\frac{4\lambda n}{r}}\right).$$

Further, since $c<0$ and $c=\frac{2\gamma}{r}b$, we have $b<0$, and since c increases monotonically in λ so does b. Using the relation that $c=\frac{2\gamma}{r}b$ and equation (91), we have that

$$\alpha_3=\frac{c-\lambda}{-2b(n-1)}=-\frac{\gamma}{r(n-1)}+\frac{\lambda}{2b(n-1)},$$

while, using (90),

$$\begin{aligned}0&=-(n-2)c-\alpha_5(n-1)b-\lambda\\ \alpha_5&=\frac{-(n-2)c-\lambda}{b(n-1)}\\ &=-\frac{n-2}{n-1}\frac{2\gamma}{r}-\frac{\lambda}{b(n-1)}\\ &=\frac{-2\gamma}{r}-2\alpha_3.\end{aligned}$$

Recall that

$$\alpha_1=\frac{1}{r+\lambda-c}(rv+\frac{ac}{b}+\lambda R_4),$$

where, based on the transfers, $R_4=\frac{-a}{b}$, so

$$\alpha_1=\frac{1}{r+\lambda-c}(rv+\frac{ac}{b}-\frac{a\lambda}{b}),$$

and from the first order condition for auction demand,

$$0=\frac{1}{b(n-1)}a+(\alpha_1+\alpha_5\frac{-a}{c})-\frac{\lambda}{b(n-1)}\frac{a}{c}.$$

Plugging in $\alpha_5=\frac{-2\gamma}{r}-2(\frac{c-\lambda}{-2b(n-1)})$,

$$0=\alpha_1+\frac{2\gamma}{r}\frac{a}{c}\Rightarrow\alpha_1=-\frac{a}{b},$$

and plugging this into the above,

$$\alpha_1 = \frac{1}{r+\lambda-c}(rv + -c\alpha_1 + \lambda\alpha_1),$$

from which it is clear that $\alpha_1 = v$ and $a = -bv$. Returning to the coefficients $\hat{a}, \hat{b}$ defining $\delta(\phi)$, since $\frac{a}{c} = -v\frac{r}{2\gamma}$ and $\frac{b}{c} = \frac{r}{2\gamma}$, we have

$$\begin{aligned}
\hat{a} &= \frac{a}{c} - \frac{1}{2n\kappa_0}(-\frac{n-1}{n}(\alpha_1 - \frac{\alpha_5 a}{c}) + \frac{2(n-1)\alpha_3 a}{nc}) \\
&= \frac{-vr}{2\gamma} - \frac{1}{2n\kappa_0}(-\frac{n-1}{n}(v - v(\frac{2\gamma}{r})(\frac{r}{2\gamma}))) \\
&= \frac{-vr}{2\gamma}, \\
\hat{b} &= \frac{b}{c} - \frac{1}{2n\kappa_0}(\frac{n-1}{n}(\frac{\alpha_5 b}{c}) + \frac{2(n-1)\alpha_3 b}{nc} + 1) \\
&= \frac{r}{2\gamma} - \frac{1}{2n^2\kappa_0}.
\end{aligned}$$

Returning to the system of value function coefficients, it remains to calculate

$$\begin{aligned}
\alpha_4 &= \frac{1}{r}(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2) \\
\alpha_2 &= \frac{1}{r}(\frac{ca}{-b} + (\lambda - c)\alpha_1 + \lambda n R_1) \\
\alpha_0^i &= \frac{1}{r}(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \lambda R_0).
\end{aligned}$$

Plugging in the equilibrium formulas for R_2, R_1, R_0,

$$\begin{aligned}
\alpha_4 &= \frac{1}{r}(\frac{c^2}{-b} + (\lambda - c)\alpha_5 + \lambda\alpha_3 + \frac{c\lambda}{b}) \\
\alpha_2 &= \frac{1}{r}(\frac{ca}{-b} + (\lambda - c)v + \frac{a\lambda}{b}) \\
\alpha_0^i &= \frac{1}{r}(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}),
\end{aligned}$$

and using the definitions of a, b, c,

$$\begin{aligned}
\alpha_4 &= \frac{1}{r}(-\frac{2\gamma}{r}c + (\lambda - c)(\frac{-2\gamma}{r} - 2\alpha_3) + \lambda\alpha_3 + \frac{c\lambda}{b}) \\
\alpha_2 &= \frac{1}{r}(cv + (\lambda - c)v + -v\lambda),
\end{aligned}$$

implying $\alpha_2 = 0$ and

$$\begin{aligned}
\alpha_4 &= \frac{1}{r}(2c\alpha_3 + \lambda(\frac{-2\gamma}{r} - 2\alpha_3) + \lambda\alpha_3 + \frac{2\gamma\lambda}{r}) \\
&= \frac{1}{r}(2c - \lambda)\alpha_3 = \frac{\gamma}{r} + \alpha_3.
\end{aligned}$$

Finally, this implies that

$$\begin{aligned}
\alpha_0^i &= \frac{1}{r}(\frac{\gamma}{r}\frac{\sigma_Z^2}{n^2} + \alpha_3(\frac{\sigma_Z^2}{n^2} + \sigma_i^2) + \alpha_5\frac{\rho^i}{n}) \\
&= \frac{1}{r}(\frac{\gamma}{r}\frac{\sigma_Z^2}{n^2} + \alpha_3(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}) - \frac{2\gamma}{r}\frac{\rho^i}{n}) \\
&= \frac{1}{r}(\frac{\gamma}{r}\frac{\sigma_Z^2}{n^2} + (-\frac{\gamma}{r(n-1)} + \frac{\lambda}{2b(n-1)})(\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}) - \frac{2\gamma}{r}\frac{\rho^i}{n}).
\end{aligned}$$

Note that $\frac{\sigma_Z^2}{n^2} + \sigma_i^2 - 2\frac{\rho^i}{n}$ is the variance of $(\frac{Z_1}{n} - H_1^i)$ conditional on Z_0 and thus positive, so α_0^i declines in λ because $b < 0$ and b increases with λ.

Finally, we must verify that in equilibrium, $\phi = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}$. We see from the definitions of a, b, c that

$$\phi = \frac{a + c\bar{Z}}{-b} = v - \frac{2\gamma}{r}\bar{Z}$$

while from the definition of α_5, α_3 we have $2\alpha_3 + \alpha_5 = \frac{-2\gamma}{r}$, so this holds with probability 1.

D.6 Finishing the Verification

In this section, we show that at the $V(z, Z)$ and strategies which solve the HJB, using any alternate admissable strategy leads to an inferior payoff for each trader. We fix some admissible demand process D^i, and report process $\tilde{z}$, by which the inventory of trader i at time t is

$$z_t^{(D,\tilde{z})} = z_0^i + \int_0^t D_s^i ds + H_t^i + \int_0^t Y^i((\tilde{z}_s, \hat{z}_s^{-i}))dN_s. \tag{94}$$

Following the steps of the derivation of the value function, we can show that under the laws of motion implied by $D^i, \tilde{z}$,

$$\begin{aligned}
\mathbb{E}[U(X_\mathcal{T}) - U(X_0)] = \mathbb{E}[\int_{0+}^{\mathcal{T}} & D_s^i(\alpha_1 + \alpha_5\bar{Z}_{s-} + 2\alpha_3 z_{s-}^{D,\tilde{z}}) + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_3\sigma_i^2 + \alpha_5\frac{\rho^i}{n} \\
& + \lambda Y^i((\tilde{z}_s, \hat{z}_s^{-i}))(\alpha_1 + \alpha_5\bar{Z}_{s-} + 2\alpha_3 z_{s-}^{D,\tilde{z}} + \alpha_3 Y^i((\tilde{z}_s, \hat{z}_s^{-i}))) + r(v z_s^{D,\tilde{z}} - V(z_s^{D,\tilde{z}}, Z_s))ds].
\end{aligned}$$

Since $\alpha_0 - \alpha_5$ satisfy the HJB, we have

$$\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] \leq \mathbb{E}[\int_{0+}^{\mathcal{T}} D_s^i \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^{D,\tilde{z}}) + \gamma (z_s^{D,\tilde{z}})^2 ds$$
$$- \int_0^{\mathcal{T}} \hat{T}^i((\hat{z}_s^i, \hat{z}_s^{-i}); \Phi_{(a,b,c)}(D_{s-}^i; Z_{s-} - z_{s-}^{D,\tilde{z}}))dN_s],$$

and rearranging, this is

$$V(z_0^i, Z_0) \geq \mathbb{E}[\pi z_{\mathcal{T}}^{D,\tilde{z}} + \int_{0+}^{\mathcal{T}} -D_s^i \Phi_{(a,b,c)}(D_s^i; Z_s - z_s^{D,\tilde{z}}) - \gamma (z_s^{D,\tilde{z}})^2 ds$$
$$+ \int_0^{\mathcal{T}} \hat{T}^i((\hat{z}_s^i, \hat{z}_s^{-i}); \Phi_{(a,b,c)}(D_{s-}^i; Z_{s-} - z_{s-}^{D,\tilde{z}}))dN_s].$$

Since this holds with equality for the conjectured linear strategy, the linear strategy is optimal.

E Proof of Proposition 6

The proof is extremely similar to proposition 4, so we leave some details to the reader. We write $V(z, Z)$ rather than $V_M^i(z, Z)$ for brevity. For any affine κ_1, κ_2 functions, the transfers in equilibrium take the form

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

for constants $R_0 - R_4$. In any symmetric equilibrium, the value function

$$V(z, Z) = \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_0^{\mathcal{T}} (-\gamma (z_s^i)^2 \, ds) + \int_0^{\mathcal{T}} T_\kappa^i(\hat{z}_s, Z_s) \, dN_s\right]$$

takes the form

$$V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z \bar{Z},$$

where

$$
\begin{aligned}
\alpha_3 &= \frac{-\gamma}{r+\lambda} \\
\alpha_5 &= \frac{1}{r+\lambda}(\lambda n R_3) \\
\alpha_4 &= \frac{1}{r}(\lambda\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2) \\
\alpha_1 &= \frac{1}{r+\lambda}(rv + \lambda R_4) \\
\alpha_2 &= \frac{1}{r}(\lambda\alpha_1 + \lambda n R_1) \\
\alpha_0^i &= \frac{1}{r}(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n} + \lambda R_0),
\end{aligned}
$$

and where R_0 through R_4 are the previously defined transfer coefficients. To see this, note that given the α coefficients, we have

$$
\begin{aligned}
&(r+\lambda)\left(\alpha_0^i + \alpha_1 z + \alpha_2\bar{Z} + \alpha_3 z^2 + \alpha_4\bar{Z}^2 + \alpha_5 z\bar{Z}\right) \\
&\quad = rvz - \gamma z^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_3\sigma_i^2 + \alpha_5\frac{\rho^i}{n} \\
&\qquad + \lambda(\alpha_0^i + \alpha_1\bar{Z} + \alpha_2\bar{Z} + \alpha_3\bar{Z}^2 \\
&\qquad + \alpha_4\bar{Z}^2 + \alpha_5\bar{Z}^2 + R_0 + R_1 Z + R_2 Z^2 + R_3 Zz + R_4 z).
\end{aligned}
$$

Let $Y_t = 1_{\{\mathcal{T}\leq t\}}$ and $V(z,Z)$ be defined as above. Let

$$
X = \begin{bmatrix} z_t^i \\ Z_t \\ Y_t \end{bmatrix}
$$

and $U(X) = U(z,Z,Y) = (1-Y)V(z,Z) + Yvz$. Then, following the steps of the proof of proposition 4, if we let

$$
\chi_s = \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_3\sigma_i^2 + \alpha_5\frac{\rho^i}{n} - \lambda(z_s^i - \bar{Z}_s)(\alpha_1 + \alpha_5\bar{Z}_{s-} + \alpha_3(z_{s-}^i + \bar{Z}_{s-})) + r(vz_s^i - V(z_s^i, Z_s)),
$$

we can show that

$$
\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left[\int_{0+}^{\mathcal{T}} \chi_s\, ds\right].
$$

Because α_0^i through α_5 satisfy the system of equations specified at the beginning of this proof, we have

$$
\mathbb{E}[U(X_{\mathcal{T}}) - U(X_0)] = \mathbb{E}\left[\int_{0+}^{\mathcal{T}} \bar{\chi}_s\, ds\right],
$$

where

$$\bar{\chi}_s = \gamma(z_s^i)^2 - \lambda(R_0 + R_1 Z_s + R_2 Z_s^2 + R_3 Z_s z_s^i + R_4 z_s^i).$$

Using the definitions of $U, \mathcal{T}$, and R_0 through R_4, as well as the fact that $\mathbb{E}[v z_{\mathcal{T}}^i] = \mathbb{E}[\pi z_{\mathcal{T}}^i]$, we can rearrange to find that

$$\begin{aligned}
V(z_0^i, Z_0) &= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0+}^{\mathcal{T}} \bar{\chi}_s \, ds\right] \\
&= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0+}^{\mathcal{T}} -\gamma(z_s^i)^2 + \lambda T_\kappa^i(\hat{z}_s, Z_s)\, ds\right] \\
&= \mathbb{E}\left[\pi z_{\mathcal{T}}^i + \int_{0}^{\mathcal{T}} -\gamma(z_s^i)^2 \, ds + \int_{0}^{\mathcal{T}} T_\kappa^i(\hat{z}_s, Z_s)\, dN_s\right],
\end{aligned}$$

which completes the proof that the value function $V(z, Z)$ takes the form above. The arguments of section C.3 go through exactly the same (with these different α coefficients), so it must be that

$$\kappa_1(Z) = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z},$$

and the equilibrium reports are optimal as long as

$$\kappa_2(Z) = \hat{a} + \hat{b}Z = -\bar{Z} - \frac{\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}}{2\kappa_0 n^2}.$$

Once again the equilibrium transfers are $(\alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z})\left(z^i - \bar{Z}\right)$, so the coefficients R_m in

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

are given by

$$\begin{aligned}
R_0 &= 0 \\
R_1 &= -\frac{\alpha_1}{n} \\
R_2 &= -\frac{\alpha_5 + 2\alpha_3}{n^2} \\
R_3 &= \frac{\alpha_5 + 2\alpha_3}{n} \\
R_4 &= \alpha_1.
\end{aligned}$$

From above we have that

$$\alpha_3 = \frac{-\gamma}{r+\lambda}$$
$$\alpha_5 = \frac{1}{r+\lambda}(\lambda n R_3)$$
$$\alpha_4 = \frac{1}{r}(\lambda\alpha_5 + \lambda\alpha_3 + \lambda n^2 R_2)$$
$$\alpha_1 = \frac{1}{r+\lambda}(rv + \lambda R_4)$$
$$\alpha_2 = \frac{1}{r}(\lambda\alpha_1 + \lambda n R_1)$$

so, plugging in R_1, R_2, R_3, R_4, and rearranging,

$$\alpha_3 = \frac{-\gamma}{r+\lambda}$$
$$\alpha_5 = \frac{1}{r}(2\lambda\alpha_3) = \frac{2\lambda}{r}(\frac{-\gamma}{r+\lambda})$$
$$\alpha_4 = \frac{1}{r}(\lambda\alpha_5 + \lambda\alpha_3 - \lambda(\alpha_5 + 2\alpha_3)) = \frac{\lambda}{r}(\frac{\gamma}{r+\lambda})$$
$$\alpha_1 = \frac{1}{r}(rv) = v$$
$$\alpha_2 = \frac{1}{r}(\lambda\alpha_1 - \lambda\alpha_1) = 0.$$

Thus, letting $\alpha_1 - \alpha_5$ be these values and

$$\alpha_0^i = \frac{1}{r}(\alpha_3\sigma_i^2 + \alpha_4\frac{\sigma_Z^2}{n^2} + \alpha_5\frac{\rho^i}{n}),$$

and defining the value function

$$V(z^i, Z) = \alpha_0^i + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3 (z^i)^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z^i \bar{Z},$$

This solves the associated HJB equation

$$0 = -\gamma(z^i)^2 + r(vz^i - V(z^i, Z)) + \frac{\sigma_i^2}{2}V_{zz}(z^i, Z) + \frac{\sigma_Z^2}{n^2}V_{ZZ}(z^i, Z) + 2\frac{\rho^i}{n}V_{zZ}(z^i, Z)$$
$$+ \sup_{\hat{z}^i} \lambda\left(V(z^i + Y^i((\hat{z}^i, \hat{z}^{-i})), Z) - V(z^i, Z) + T_\kappa^i((\hat{z}^i, \hat{z}^{-i}), Z)\right).$$

Plugging in $\alpha_1, \alpha_3, \alpha_5$, we have

$$\kappa_1(Z) = v - \frac{2\gamma}{r}\bar{Z},$$

$$\kappa_2(Z) = -\bar{Z} - \frac{v - \frac{2\gamma}{r}\bar{Z}}{2\kappa_0 n^2}.$$

The last part of the verification, demonstrating that alternate strategies do weakly worse, is exactly the same as in proposition 4 and thus omitted. Rearranging the $\alpha_0^i - \alpha_5$ above gives the expression in proposition 6, completing the proof.

F The Impaired Mechanism

In this section, we consider an alternate mechanism designed to reduce a fraction ξ of the excess inventory at each implementation. Its allocations and transfers are given by

$$Y^i(\hat{z}) = \xi \left(\frac{\sum_j \hat{z}^j}{n} - \hat{z}^i \right) \tag{95}$$

$$\begin{aligned} T^i(\hat{z}, Z) = {} & \kappa_0 (n\kappa_2(Z) + \xi \sum_j \hat{z}^j)^2 + \kappa_1(Z)(\xi \hat{z}^i + \kappa_2(Z)) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} \\ & + n\kappa_0 \frac{1-\xi}{\xi} [(\xi \hat{z}^i + \kappa_2(Z))^2 - ((n-1)\kappa_2(Z) + \xi \sum_{j \neq i} \hat{z}^j + \frac{\xi \kappa_1(Z)}{2\kappa_0 n})^2], \end{aligned}$$

for a constant $\kappa_0 < 0$ and affine functions $\kappa_1(Z), \kappa_2(Z)$. It is worth noting that the sum of these transfers may not be weakly negative for any reports $\hat{z}$, but we show in all the equilibria we consider the transfers sum to zero with probability 1.

F.1 Proof Sketch for Alternate Proposition 4

We provide a sketch of a proof for an alternative version of proposition 4: for any $\xi \in (0, 1]$, there will exist a unique symmetric equilibrium such that, each time the mechanism is run, all traders reduce a fraction ξ of their inventory imbalance $z^i - \bar{Z}$. The auction price and value functions are identical, and the auction demands are identical replacing λ with $\lambda(2\xi - \xi^2)$. The mechanism demands are still $\hat{z}^i = z^i$.

Proof sketch: In any such equilibrium, each trader reports $\hat{z}^i = z^i$, such that

$$Y^i(\hat{z}_t) = \xi(\bar{Z} - z_t^i)$$

and the transfers are

$$\begin{aligned}
T^i(\hat{z}, Z) &= \kappa_0(n\kappa_2(Z) + \xi Z)^2 + \kappa_1(Z)(\xi z^i + \kappa_2(Z)) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} \\
&\quad + n\kappa_0 \frac{1-\xi}{\xi}[(\xi z^i + \kappa_2(Z))^2 - ((n-1)\kappa_2(Z) + \xi(Z - z^i) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n})^2] \\
&= \kappa_0(n\kappa_2(Z) + \xi Z)^2 + \kappa_1(Z)(\xi z^i + \kappa_2(Z)) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} \\
&\quad + n\kappa_0 \frac{1-\xi}{\xi}\left(\xi Z + n\kappa_2(Z) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right)\left(\xi z^i + \kappa_2(Z) - ((n-1)\kappa_2(Z) + \xi(Z - z^i) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n})\right).
\end{aligned}$$

For any affine κ_1, κ_2, it follows that the transfer will be given by

$$R_0 + R_1 Z_t + R_2 Z_t^2 + R_3 Z_t z_t^i + R_4 z_t^i,$$

for constants $R_0 - R_4$. Receiving such transfers at Poisson arrival times must lead to a linear-quadratic value function, as in the proofs the previous propositions. That is, the equilibrium continuation value function V for agent i must be

$$V(z^i, Z) = \alpha_0^i + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3 (z^i)^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z^i \bar{Z}. \tag{96}$$

Fix reports $\hat{z}^j = z^j$ for the other traders. When trader i chooses $\tilde{z}$, they maximize

$$(\alpha_1 + \alpha_5 \bar{Z})Y^i((\tilde{z}, \hat{z}^{-i})) + \alpha_3 Y^i((\tilde{z}, \hat{z}^{-i}))^2 + 2\alpha_3 Y^i((\tilde{z}, \hat{z}^{-i}))z^i + T^i((\tilde{z}, \hat{z}^{-i}), Z),$$

where, writing $\kappa_2(Z) = \hat{a} + \hat{b}Z$ and $\hat{z}^j = z^j$,

$$\begin{aligned}
T^i((\tilde{z}, \hat{z}^{-i}), Z) &= \kappa_0(\xi\tilde{z} + n\hat{a} + n\hat{b}Z + \xi(Z - z^i))^2 + \kappa_1(Z)(\xi\tilde{z} + \hat{a} + \hat{b}Z) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} \\
&\quad + n\kappa_0 \frac{1-\xi}{\xi}[(\xi\hat{z}^i + \hat{a} + \hat{b}Z)^2 - ((n-1)(\hat{a} + \hat{b}Z) + \xi\sum_{j\neq i}\hat{z}^j + \frac{\xi\kappa_1(Z)}{2\kappa_0 n})^2].
\end{aligned}$$

Taking a first order condition,

$$\begin{aligned}
&-\frac{n-1}{n}\xi(\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3 z^i) - \frac{2(n-1)\alpha_3\xi}{n}Y^i((\tilde{z}, \hat{z}^{-i})) + \xi\kappa_1(Z) \\
&+ 2\kappa_0\xi(\xi\tilde{z} + n\hat{a} + n\hat{b}Z + \xi(Z - z^i)) + 2n\kappa_0\xi\frac{1-\xi}{\xi}(\xi\hat{z}^i + \hat{a} + \hat{b}Z) = 0.
\end{aligned}$$

Plugging in $\tilde{z} = z^i$, $Y^i((\tilde{z}, \hat{z}^{-i})) = \xi(\bar{Z} - z^i)$, and dividing through by ξ, we have

$$
\begin{aligned}
&-\frac{n-1}{n}(\alpha_1+\alpha_5\bar{Z}+2\alpha_3 z^i)-\frac{2(n-1)\alpha_3}{n}\xi(\bar{Z}-z^i)+\kappa_1(Z)\\
&+2\kappa_0(n\hat{a}+n\hat{b}Z+\xi Z)+2n\kappa_0\frac{1-\xi}{\xi}(\xi z^i+\hat{a}+\hat{b}Z)=0.
\end{aligned}
$$

It is clear that the z^i terms will cancel if and only if $\kappa_0=(n-1)\alpha_3/n^2$. Given this, the unique $\hat{a},\hat{b}$ solving this is given by

$$
\begin{aligned}
0=&-\frac{n-1}{n}(\alpha_1+\alpha_5\bar{Z})-\frac{2(n-1)\alpha_3}{n}\xi(\bar{Z})+\kappa_1(Z)\\
&+\frac{2(n-1)\alpha_3}{n^2}(n\hat{a}+n\hat{b}Z+\xi Z))+\frac{2(n-1)\alpha_3}{n}\frac{1-\xi}{\xi}(\hat{a}+\hat{b}Z),
\end{aligned}
$$

$$
\begin{aligned}
\hat{a}+\hat{b}Z&=\frac{n\xi}{2(n-1)\alpha_2}\left(-\kappa_1(Z)+(\alpha_1+\alpha_5\bar{Z})\frac{n-1}{n}\right)\\
&=\frac{\xi}{2n\kappa_0}\left(-\kappa_1(Z)+(\alpha_1+\alpha_5\bar{Z})\frac{n-1}{n}\right).
\end{aligned}
$$

Manipulating the formula for transfers, we can write the equilibrium transfer for trader i, given $\hat{z}^i=z^i$ for all i, as

$$
\begin{aligned}
&=\kappa_0(n\kappa_2(Z)+\xi Z)^2+\kappa_1(Z)(\xi z^i+\kappa_2(Z))+\frac{(2\xi-\xi^2)\kappa_1^2(Z)}{4n^2\kappa_0}\\
&+n\kappa_0\frac{1-\xi}{\xi}\left(\xi Z+n\kappa_2(Z)+\frac{\xi\kappa_1(Z)}{2\kappa_0 n}\right)\left(\xi z^i+\kappa_2(Z)-((n-1)\kappa_2(Z)+\xi(Z-z^i)+\frac{\xi\kappa_1(Z)}{2\kappa_0 n})\right).
\end{aligned}
$$

Suppose we define κ_1 such that

$$
\xi Z+n\kappa_2(Z)+\frac{\xi\kappa_1(Z)}{2\kappa_0 n}=0.
$$

Then this simplifies to

$$
\kappa_0(n\kappa_2(Z)+\xi Z)^2+\kappa_1(Z)(\xi z^i+\kappa_2(Z))+\frac{(2\xi-\xi^2)\kappa_1^2(Z)}{4n^2\kappa_0},
$$

and summing across traders, this is

$$
n\kappa_0(\frac{\xi\kappa_1(Z)}{2\kappa_0 n})^2-\kappa_1(Z)(\frac{\xi\kappa_1(Z)}{2\kappa_0 n})+\frac{(2\xi-\xi^2)\kappa_1^2(Z)}{4n\kappa_0}=0.
$$

Some calculation shows that the above κ_1 is the unique one such that the transfers sum to zero with probability 1, which must be the case for IR and budget balance to hold. Plugging in the formula for κ_2, we see we need

$$\begin{aligned}
0 &= \xi Z + \frac{\xi}{2\kappa_0}\left(-\kappa_1(Z) + (\alpha_1 + \alpha_5 \bar{Z})\frac{n-1}{n}\right) + \frac{\xi\kappa_1(Z)}{2\kappa_0 n} \\
0 &= 2\kappa_0 n Z + \big(-n\kappa_1(Z) + (\alpha_1 + \alpha_5\bar{Z})(n-1)\big) + \kappa_1(Z) \\
\kappa_1(Z) &= (\alpha_1 + \alpha_5\bar{Z}) + \frac{2\kappa_0 n}{n-1} Z \\
&= \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}.
\end{aligned}$$

This is the unique $\kappa_1(Z)$ consistent with budget balance and ex-post IR. The HJB equation is

$$\begin{aligned}
rV(z^i, Z) = &-\gamma(z^i)^2 + rvz + \frac{\sigma_i^2}{2}V_{zz}(z^i, Z) + \frac{\sigma_Z^2}{n^2}V_{ZZ}(z^i, Z) + 2\frac{\rho^i}{n}V_{zZ}(z^i, Z) \\
&+ \sup_{D,\tilde{z}} -\Phi_{(a,b,c)}(D; Z - z^i)D + V_z(z^i, Z)D + \lambda\left(V(z^i + Y^i(\tilde{z}, \hat{z}^{-i}), Z) - V(z^i, Z) + T^i((\tilde{z}, \hat{z}^{-i}), Z)\right).
\end{aligned}$$

We just showed that given V is quadratic, so at the unique candidate equilibrium reallocations,

$$V(z + Y^i(\tilde{z}, \hat{z}^{-i}), Z) - V(z, Z) = (\alpha_1 + \alpha_5\bar{Z})\xi(\bar{Z} - z) + \alpha_3\xi^2(\bar{Z} - z)^2 + 2\alpha_3\xi z(\bar{Z} - z).$$

By the above, the equilibrium transfer is

$$\begin{aligned}
&\kappa_0(\frac{\xi\kappa_1(Z)}{2\kappa_0 n})^2 + \kappa_1(Z)(\xi(z^i - \bar{Z}) - \frac{\xi\kappa_1(Z)}{2\kappa_0 n}) + \frac{(2\xi - \xi^2)\kappa_1^2(Z)}{4n^2\kappa_0} \\
&= \kappa_1(Z)\xi(z^i - \bar{Z}).
\end{aligned}$$

Plugging in $\kappa_1(Z) = \alpha_1 + (\alpha_5 + 2\alpha_3)\bar{Z}$ and summing the transfer and the change in continuation value, this is

$$\begin{aligned}
&(\alpha_1 + \alpha_5\bar{Z})\xi(\bar{Z} - z) + \alpha_3\xi^2(\bar{Z} - z)^2 + 2\alpha_3\xi z(\bar{Z} - z) - (\alpha_1 + \alpha_5\bar{Z} + 2\alpha_3\bar{Z})\xi(\bar{Z} - z) \\
&\qquad = \alpha_3\xi^2(\bar{Z} - z)^2 - 2\alpha_3\xi\left(z^2 + \bar{Z}^2 - 2z\bar{Z}\right) \\
&\qquad = -\alpha_3(2\xi - \xi^2)(\bar{Z} - z)^2.
\end{aligned}$$

Plugging this in, the HJB becomes

$$\begin{aligned}
rV(z^i, Z) = &-\gamma(z^i)^2 + rvz^i + \frac{\sigma_i^2}{2}V_{zz}(z^i, Z) + \frac{\sigma_Z^2}{n^2}V_{ZZ}(z^i, Z) + 2\frac{\rho^i}{n}V_{zZ}(z^i, Z) \\
&+ \sup_{D} -\Phi_{(a,b,c)}(D; Z - z^i)D + V_z(z^i, Z)D - \lambda(2\xi - \xi^2)\alpha_3\left(z^i - \bar{Z}\right)^2.
\end{aligned}$$

This is exactly the HJB from the proof of proposition 4, replacing λ with $\lambda^* = \lambda(2\xi - \xi^2)$.

G Discrete Time Results

In this appendix, we analyze discrete time versions of the models of sections 3, 4, and 5. The focus is the existence of a subgame perfect equilibrium in each complete information game, which corresponds to a Perfect Bayes equilibrium of each incomplete information game. We also show convergence results for the models of sections 3 and 4. All the results are presented informally, with focus on the calculation of the equilibrium, but these arguments can all be made fully rigorous.

The primitive setting, other than mechanisms, is identical to Duffie and Zhu (2017). Specifically, $n > 2$ traders trade in each period $k \in \{0, 1, 2, ...\}$, where trading periods are separated by clock time Δ so that the k-th auction occurs at time $k\Delta$.

In each period k, each trader i submits an auction order $x_{ik}(p_k)$ for how many units of asset they wish to purchase if the auction price is p_k. We focus on affine equilibria in which each trader chooses

$$x_{ik}(p_k) = a + bp_k + cz_{ik},$$

where z_{ik} is trader i's inventory entering period k, for constants a, c and $b \neq 0$. If $n-1$ traders use such a strategy with the same constants a, b, c, then there is a unique market clearing price $\Phi_{(a,b,c)}(D, Z - z)$ for any demand D submitted by trader i, which is given by

$$\Phi_{(a,b,c)}(D, Z - z) = \frac{(n-1)a + c(Z_k - z_{ik}) + D}{-b(n-1)}.$$

Each trader also submits a contingent mechanism report $\hat{z}_{ik}(p_k)$. With probability q, a mechanism occurs: each trader receives a net reallocation

$$Y^i(\hat{z}) = \frac{\sum_{j=1}^{n} \hat{z}_{jk}}{n} - \hat{z}_{ik}$$

and a transfer which will be described shortly and might depend upon p_k. With probability $1-q$, a double auction occurs, and each trader receives $x_{ik}(p_k)$ units of asset at a cost $p_k x_{ik}(p_k)$. If trader i ends period k with inventory z_{ik}^+, then in between periods k and $k+1$, they receive flow expected utility

$$-\frac{\gamma}{r}(1 - e^{-r\Delta})(z_{ik}^+)^2 + v(1 - e^{-r\Delta})(z_{ik}^+)$$

which can be motivated as in Duffie and Zhu. Let $\mathbf{1}_{M^k}$ equal 1 if and only if a mechanism occurs in period k, and let $\mathbf{1}_{M^k}^c = 1 - \mathbf{1}_{M^k}$. Then, in any equilibrium in which mechanisms implement efficient allocations, the equilibrium inventory evolves as

$$z_{i,k+1} = w_{i,k+1} + \mathbf{1}_{M^k}\bar{Z}_k + \mathbf{1}_{M^k}^c\left((1+c)z_{i,k} - c\bar{Z}_k\right)$$

where $w_{i,k+1}$ is an i.i.d zero mean finite variance random variable.

G.1 Observable Z

Suppose the aggregate Z_k is observable and the transfers are given by

$$T^i_\kappa(\hat{z}, Z) = \kappa_0(n\kappa_2(Z_k) + \sum_j \hat{z}_{jk})^2 + \kappa_1(Z_k)(\hat{z}_{ik} + \kappa_2(Z_k)) + \frac{\kappa_1(Z_k)^2}{4\kappa_0 n^2}.$$

Just as in the continuous time proof, at the equilibrium reports for affine κ_1, κ_2, this must take the form

$$R_0 + R_1 Z_k + R_2 Z_k^2 + R_3 Z_k z_{ik} + R_4 z_{ik}.$$

We solve for a subgame perfect equilibrium in which all traders submit

$$x_{ik}(p_k) = a + bp_k + cz_{ik},$$

$$\hat{z}_{ik}(p_k) = z_{ik}.$$

In such an equilibrium, the continuation value $V(z, Z)$ must be linear quadratic. Specifically, the continuation value is

$$\begin{aligned} V(z, Z) = \mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k}[q \left(R_0 + R_1 Z_k + R_2 Z_k^2 + R_3 Z_k z_{ik} + R_4 z_{ik} - \frac{\gamma}{r}(1 - e^{-r\Delta})(\bar{Z}_k)^2 + v(1 - e^{-r\Delta})(\bar{Z}_k)\right) \\ + (1 - q) \left(-x_{ik}(p_k)p_k - \frac{\gamma}{r}(1 - e^{-r\Delta})(x_{ik}(p_k) + z_{ik})^2 + v(1 - e^{-r\Delta})(x_{ik}(p_k) + z_{ik})\right)]] \end{aligned}$$

Given $z_{i0} = z, Z_0 = Z$ and

$$\sum_i x_{ik} p_k = 0$$

$$z_{i,k+1} = w_{i,k+1} + \mathbf{1}_{M^k} \left(z_{ik} + \frac{\sum_{j=1}^n \hat{z}_{jk}}{n} - \hat{z}_{ik} \right) + \mathbf{1}^c_{M^k} \left(z_{ik} + x_{ik}(p_k) \right).$$

Fix the conjectured equilibrium a, b, c with truthtelling ($\hat{z}_{ik} = z_{ik}$), so that

$$z_{i,k+1} = w_{i,k+1} + \mathbf{1}_{M^k} \bar{Z}_k + \mathbf{1}^c_{M^k} \left((1 + c) z_{i,k} - c\bar{Z}_k \right). \tag{97}$$

The expression for $V(z, Z)$ can be decomposed into a linear combination of discounted sums of moments of z_{ik}, Z_k. We calculate these now. Straightforward calculation shows

$$\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} Z_k] = \frac{Z_0}{1-e^{-r\Delta}} = S_0 Z_0$$
$$\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} Z_k^2] = \frac{Z_0^2}{1-e^{-r\Delta}} + \frac{\sigma_Z^2 e^{-r\Delta}}{1-e^{-r\Delta}} = S_0 Z_0^2 + S_1,$$

where $\sigma_Z^2 \equiv Var(\sum_i w_{i,k+1})$. Subtracting $\bar{Z}_{i,k+1}$ from both sides of equation (97), rearranging, and taking an expectation gives

$$\mathbb{E}[z_{i,k+1} - \bar{Z}_{k+1}] = (1-q)(1+c)\mathbb{E}[z_{i,k} - \bar{Z}_k].$$

Some calculation then shows

$$\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}] = \frac{z_{i0} - \bar{Z}_0}{1-e^{-r\Delta}(1+c)(1-q)} + \frac{\bar{Z}_0}{1-e^{-r\Delta}} = S_2(z_{i0} - \bar{Z}_0) + S_0 \bar{Z}_0,$$

as long as $|e^{-r\Delta}(1+c)(1-q)| < 1$. Subtracting $\bar{Z}_{i,k+1}$ from both sides of equation (97), then multiplying both sides by $\bar{Z}_{i,k+1}$, and taking an expectation gives

$$E[z_{i,k+1} Z_{k+1} - \bar{Z}_{k+1}^2] = (\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2}) + (1-q)(1+c)E[z_{i,k}\bar{Z}_k - \bar{Z}_k^2],$$

where $\rho^i = \mathbb{E}[w_{i,k+1}(\sum_i w_{i,k+1})]$. Then we see that

$$\begin{aligned}
\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\bar{Z}_k] &= \mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} \left(z_{ik}\bar{Z}_k - \bar{Z}_k^2\right)] + S_0 \bar{Z}_0^2 + \frac{S_1}{n^2} \\
&= z_{i0}\bar{Z}_0 - \bar{Z}_0^2 + e^{-r\Delta} \sum_{k=1}^{\infty} e^{-r\Delta(k-1)} \mathbb{E}[z_{ik}\bar{Z}_k - \bar{Z}_k^2] + S_0 \bar{Z}_0^2 + \frac{S_1}{n^2} \\
&= z_{i0}\bar{Z}_0 - \bar{Z}_0^2 + e^{-r\Delta}\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} \left((\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2}) + (1-q)(1+c)E[z_{i,k}\bar{Z}_k - \bar{Z}_k^2] \right) \\
&+ S_0 \bar{Z}_0^2 + \frac{S_1}{n^2} \\
&= z_{i0}\bar{Z}_0 - \bar{Z}_0^2 + \frac{e^{-r\Delta}(\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2})}{1-e^{-r\Delta}} + (1 - e^{-r\Delta}(1-q)(1+c))(S_0 \bar{Z}_0^2 + \frac{S_1}{n^2}) \\
&+ (1-q)(1+c)e^{-r\Delta}\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\bar{Z}_k],
\end{aligned}$$

and rearranging delivers

$$\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\bar{Z}_k] = S_0\bar{Z}_0^2 + \frac{S_1}{n^2} + \frac{z_{i0}\bar{Z}_0 - \bar{Z}_0^2 + \frac{e^{-r\Delta}(\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2})}{1-e^{-r\Delta}}}{1-(1-q)(1+c)e^{-r\Delta}} = S_2 z_{i0}\bar{Z}_0 + (S_0 - S_2)\bar{Z}_0^2 + S_3.$$

Finally, squaring both sides of equation (97) and taking an expectation shows that

$$\mathbb{E}[\big(z_{i,k+1} - \bar{Z}_{k+1}\big)^2] = (\frac{\sigma_Z^2}{n^2} - 2\frac{\rho^i}{n} + \sigma_i^2) + (1-q)(1+c)^2\mathbb{E}[\big(z_{i,k} - \bar{Z}_k\big)^2],$$

where $\sigma_i^2 = \mathbb{E}[w_{i,k+1}^2]$. Then

$$\sum_{k=0}^{\infty} e^{-r\Delta}\mathbb{E}[\big(z_{i,k} - \bar{Z}_k\big)^2] = \frac{\big(z_{i,0} - \bar{Z}_0\big)^2 + \frac{(\frac{\sigma_Z^2}{n^2} - 2\frac{\rho^i}{n} + \sigma_i^2)e^{-r\Delta}}{1-e^{-r\Delta}}}{1 - e^{-r\Delta}(1-q)(1+c)^2} = S_4\left(z_{i,0} - \bar{Z}_0\right)^2 + S_5,$$

as long as $|S_4^{-1}| = |1 - e^{-r\Delta}(1-q)(1+c)^2| < 1$. It follows that

$$\sum_{k=0}^{\infty} e^{-r\Delta}\mathbb{E}[z_{i,k}^2] = S_4\left(z_{i,0} - \bar{Z}_0\right)^2 + S_5 + 2\left(S_2 z_{i0}\bar{Z}_0 + (S_0 - S_2)\bar{Z}_0^2 + S_3\right) - \left(S_0\bar{Z}_0^2 + \frac{S_1}{n^2}\right).$$

In summary, letting

$$\begin{aligned}
S_0 &= \frac{1}{1-e^{-r\Delta}} \\
S_1 &= \frac{\sigma_Z^2 e^{-r\Delta}}{1-e^{-r\Delta}} \\
S_2 &= \frac{1}{1-e^{-r\Delta}(1-q)(1+c)} \\
S_3 &= S_2\frac{e^{-r\Delta}(\frac{\rho^i}{n} - \frac{\sigma_Z^2}{n^2})}{1-e^{-r\Delta}} \\
S_4 &= \frac{1}{1-e^{-r\Delta}(1-q)(1+c)^2} \\
S_5 &= S_4\frac{(\frac{\sigma_Z^2}{n^2} - 2\frac{\rho^i}{n} + \sigma_i^2)e^{-r\Delta}}{1-e^{-r\Delta}}
\end{aligned}$$

and assuming $|S_2^{-1}|, |S_4^{-1}|$ are strictly less than 1,

$$\begin{aligned}
\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}] &= S_2(z_{i0} - \bar{Z}_0) + S_0 \bar{Z}_0, \\
\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik} \bar{Z}_k] &= S_2 z_{i0} \bar{Z}_0 + (S_0 - S_2)\bar{Z}_0^2 + S_3 \\
\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} \bar{Z}_k] &= S_0 \bar{Z}_0 \\
\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} \bar{Z}_k^2] &= S_0 \bar{Z}_0^2 + \frac{S_1}{n^2}, \\
\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta} z_{i,k}^2] &= S_4 \left(z_{i,0} - \bar{Z}_0\right)^2 + S_5 + 2\left(S_2 z_{i0} \bar{Z}_0 + (S_0 - S_2)\bar{Z}_0^2 + S_3\right) - \left(S_0 \bar{Z}_0^2 + \frac{S_1}{n^2}\right).
\end{aligned}$$

Suppose that

$$V(z, Z) = \alpha_0^i + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z \bar{Z}.$$

Then the utility for having inventory z, Z immediately after an auction or mechanism is

$$\begin{aligned}
V^+(z, Z) &= -\frac{\gamma}{r}(1 - e^{-r\Delta})(z)^2 + v(1 - e^{-r\Delta})z + \mathbb{E}[e^{-r\Delta} V(z + w_{i,k+1}, Z + \sum_i w_{i,k+1})] \\
&= -\frac{\gamma}{r}(1 - e^{-r\Delta})(z)^2 + v(1 - e^{-r\Delta})z \\
&+ e^{-r\Delta}\left(\alpha_0^i + \alpha_3 \sigma_i^2 + \alpha_4 \frac{\sigma_Z^2}{n^2} + \alpha_5 \frac{\rho^i}{n} + \alpha_1 z + \alpha_2 \bar{Z} + \alpha_3 z^2 + \alpha_4 \bar{Z}^2 + \alpha_5 z \bar{Z}\right) \\
&= u(Z) + (e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1 - e^{-r\Delta}))(z - \bar{Z})^2 + (v(1 - e^{-r\Delta}) + e^{-r\Delta}\alpha_1)z \\
&+ \left(e^{-r\Delta}\alpha_5 + 2(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1 - e^{-r\Delta}))\right) z\bar{Z}.
\end{aligned}$$

We have thus shown the continuation value maximized in the mechanism takes the form of section 2, with

$$\begin{aligned}
\beta_0 &= (v(1 - e^{-r\Delta}) + e^{-r\Delta}\alpha_1) \\
\beta_1 &= \left(e^{-r\Delta}\alpha_5 + 2(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1 - e^{-r\Delta}))\right).
\end{aligned}$$

Transfers in the mechanism thus must be run with $\kappa_1(Z_k) = \beta_0 + \beta_1 \bar{Z}_k$ to be IR. From proposition 1, in the equilibrium of the mechanism game we seek (with observable Z), each trader submits $\hat{z}_{ik} = z_{ik}$ as long as

$$\kappa_2(Z_k) = -\bar{Z}_k + \frac{-(\beta_0 + \beta_1 \bar{Z}_k)}{2\kappa_0 n^2},$$

so that the sum is

$$n\kappa_2(Z_k) + \sum_i \hat{z}_{ik} = \frac{-(\beta_0 + \beta_1 \bar{Z}_k)}{2\kappa_0 n}.$$

Returning to the continuation value, in equilibrium at each mechanism event all traders receive a transfer equal to $\kappa_1(Z_k)(z_{ik} - \bar{Z}) = \left(\beta_0 + \beta_1 \bar{Z}_k\right)(z_{ik} - \bar{Z})$. The equilibrium price must equal $p_k = (a + c\bar{Z})/-b$ and the equilibrium demand $x_{ik} = c(z_{ik} - \bar{Z}_k)$. Thus, plugging in, the candidate equilibrium continuation value is

$$\begin{aligned}
V(z, Z) = \mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k}[q\left((\beta_0 + \beta_1 \bar{Z}_k)(z_{ik} - \bar{Z}_k) - \frac{\gamma}{r}(1 - e^{-r\Delta})(\bar{Z}_k)^2 + v(1 - e^{-r\Delta})(\bar{Z}_k)\right) \\
+ (1-q)\left(-c(z_{ik} - \bar{Z}_k)\frac{a + c\bar{Z}_k}{-b} - \frac{\gamma}{r}(1 - e^{-r\Delta})((1+c)z_{ik} - c\bar{Z}_k)^2\right) \\
+ (1-q)\left(v(1 - e^{-r\Delta})((1+c)z_{ik} - c\bar{Z}_k)\right)]].
\end{aligned}$$

Collecting terms,

$$\begin{aligned}
V(z, Z) = &\left(q\beta_0 + (1-q)[\frac{ca}{b} + v(1 - e^{-r\Delta})(1+c)]\right)\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}] \\
&+ \left(q\beta_1 + (1-q)[\frac{c^2}{b} + 2\frac{\gamma}{r}(1 - e^{-r\Delta})(1+c)c]\right)\mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}\bar{Z}_k] \\
&- \frac{\gamma}{r}(1 - e^{-r\Delta})(1-q)(1+c)^2 \mathbb{E}[\sum_{k=0}^{\infty} e^{-r\Delta k} z_{ik}^2] \\
&+ \epsilon(Z).
\end{aligned}$$

Plugging in definitions above, it follows that

$$\begin{aligned}
\alpha_1 &= S_2\left(q\beta_0 + (1-q)[\frac{ca}{b} + v(1 - e^{-r\Delta})(1+c)]\right) \\
\alpha_3 &= -\frac{\gamma}{r}(1 - e^{-r\Delta})(1-q)(1+c)^2 S_4 \\
\alpha_5 &= S_2\left(q\beta_1 + (1-q)[\frac{c^2}{b} + 2\frac{\gamma}{r}(1 - e^{-r\Delta})(1+c)c]\right) - \frac{\gamma}{r}(1 - e^{-r\Delta})(1-q)(1+c)^2(2(S_2 - S_4)).
\end{aligned}$$

Recalling the expressions for β_0, S_2, the α_1 equation implies

$$\begin{aligned}\beta_0 &= v(1-e^{-r\Delta}) + e^{-r\Delta}\alpha_1 \\ &= v(1-e^{-r\Delta}) + \frac{e^{-r\Delta}}{1-e^{-r\Delta}(1-q)(1+c)}\left(q\beta_0 + (1-q)[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)]\right),\end{aligned}$$

so, conjecturing and later verifying that $1-e^{-r\Delta}(1-q)(1+c)-qe^{-r\Delta} \neq 0$,

$$\beta_0 = \left(\frac{1-e^{-r\Delta}(1-q)(1+c)}{1-e^{-r\Delta}(1-q)(1+c)-qe^{-r\Delta}}\right)\left(v(1-e^{-r\Delta}) + \frac{e^{-r\Delta}(1-q)}{1-e^{-r\Delta}(1-q)(1+c)}[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)]\right).$$

A similar calculation shows that

$$\begin{aligned}\beta_1 &= e^{-r\Delta}S_2 q\beta_1 + e^{-r\Delta}S_2\left((1-q)[\frac{c^2}{b} + 2\frac{\gamma}{r}(1-e^{-r\Delta})(1+c)c]\right) \\ &- \frac{e^{-r\Delta}\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2(2(S_2-S_4)) + 2(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1-e^{-r\Delta})).\end{aligned}$$

and thus

$$\begin{aligned}\beta_1 = &\left(\frac{1-e^{-r\Delta}(1-q)(1+c)}{1-e^{-r\Delta}(1-q)(1+c)-qe^{-r\Delta}}\right) \times [e^{-r\Delta}S_2\left((1-q)[\frac{c^2}{b} + 2\frac{\gamma}{r}(1-e^{-r\Delta})(1+c)c]\right) \\ &- \frac{e^{-r\Delta}\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2(2(S_2-S_4)) + 2(e^{-r\Delta}\alpha_3 - \frac{\gamma}{r}(1-e^{-r\Delta}))].\end{aligned}$$

Putting this all together, the continuation value for trader i in a symmetric equilibrium, immediately after an auction or mechanism is run, is

$$V^+(z,Z) = u(Z) + \left(-\frac{\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2 S_4 e^{-r\Delta} - \frac{\gamma}{r}(1-e^{-r\Delta})\right)(z-\bar{Z})^2 + (\beta_0+\beta_1\bar{Z})(z-\bar{Z}).$$

Plugging in the definition of S_4, this simplifies slightly to

$$V^+(z,Z) = u(Z) + \frac{-\frac{\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(z-\bar{Z})^2 + (\beta_0+\beta_1\bar{Z})(z-\bar{Z}).$$

Trader i can choose any quantity x to purchase a price

$$\Phi(x) = \frac{1}{-b(n-1)}\left((n-1)a + c(Z-z) + x\right).$$

With observable Z, their order x is irrelevant to their payoff and continuation in the event of a mechanism. They thus maximize

$$-x\frac{1}{-b(n-1)}\left((n-1)a+c(Z-z)+x\right)+V^{+}(z+x,Z)$$

Differentiate with respect to x:

$$-\Phi(x)+\frac{x}{b(n-1)}+(\beta_0+\beta_1\bar{Z})-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(z+x-\bar{Z}),$$

and this must equal 0 with $\Phi=\phi$, $\bar{Z}=\frac{-a-b\phi}{c}$, and $x=a+b\phi+cz$. The second order condition is met if and only if $b<0$. This also implies $x=c(z-\bar{Z})$, so

$$(z+x-\bar{Z})=(1+c)z+(1+c)\frac{a+b\phi}{c}.$$

Plugging this in and gathering coefficients on $\phi, z, 1$,

$$\begin{aligned}
0&=-1+\frac{1}{n-1}-\frac{b\beta_1}{c}-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\frac{b}{c}\\
0&=\frac{c}{b(n-1)}-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\\
0&=\frac{a}{b(n-1)}+(\beta_0-\frac{a}{c}\beta_1)-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\frac{a}{c}.
\end{aligned}$$

We seek $a, b, c, \beta_1, \beta_0$ such that these three equations and the two equations defining β_0, β_1 all hold. Let ω be the larger root of

$$e^{-r\Delta}\omega^2+(n-1)(1-e^{-r\Delta})\omega-1=0,$$

so

$$\omega=\frac{-(n-1)(1-e^{-r\Delta})+\sqrt{(n-1)^2(1-e^{-r\Delta})^2+4e^{-r\Delta}}}{2e^{-r\Delta}}.$$

Then in Duffie and Zhu, when $q=0$, we can set $a=\frac{rv}{2\gamma}(1-\omega)$, $b=-\frac{r}{2\gamma}(1-\omega)$, and $c=-(1-\omega)$, and see that

$$\frac{(1+c)(1-e^{-r\Delta})}{1-e^{-r\Delta}(1+c)^2}=\frac{\frac{1-e^{-r\Delta}\omega^2}{n-1}}{1-e^{-r\Delta}\omega^2}=\frac{1}{n-1}.$$

It follows the above system holds with $\beta_0=v$, $\beta_1=\frac{-2\gamma}{r}$. Now, let $\hat{\omega}$ be the larger root of

$$e^{-r\Delta}(1-q)\hat{\omega}^2+(n-1)(1-e^{-r\Delta})\hat{\omega}-1=0,$$

so

$$\hat{\omega}=\frac{-(n-1)(1-e^{-r\Delta})+\sqrt{(n-1)^2(1-e^{-r\Delta})^2+4(1-q)e^{-r\Delta}}}{2(1-q)e^{-r\Delta}}.$$

This implies that, letting a, b, c be as before but replacing ω with $\hat{\omega}$,

$$\frac{(1+c)(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2} = \frac{\frac{1-e^{-r\Delta}(1-q)\hat{\omega}^2}{n-1}}{1-e^{-r\Delta}(1-q)\hat{\omega}^2} = \frac{1}{n-1}.$$

It is straightforward to show that a, b, c defined with $\hat{\omega}$, and $\beta_0 = v, \beta_1 = \frac{-2\gamma}{r}$ once again solve the above system. We now must verify that they satisfy the definitions of β_0, β_1. Note that under the conjectured values,

$$\begin{aligned} q\beta_0 + (1-q)[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)] &= v\left(q + (1-q)[-(1+c) + 1 + (1-e^{-r\Delta})(1+c)]\right) \\ &= v\left(1 - e^{-r\Delta}(1+c)(1-q)\right), \end{aligned}$$

from which it can be seen that $\beta_0 = v$ is consistent with the earlier system. We noted above that

$$\left(-\frac{\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2 S_4 e^{-r\Delta} - \frac{\gamma}{r}(1-e^{-r\Delta})\right) = \frac{-\frac{\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}.$$

Plugging this into the definition of β_1, we have

$$\begin{aligned} \beta_1 = &\left(\frac{1-e^{-r\Delta}(1-q)(1+c)}{1-e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}}\right) \times [e^{-r\Delta} S_2 \left((1-q)[\frac{c^2}{b} + 2\frac{\gamma}{r}(1-e^{-r\Delta})(1+c)c]\right) \\ &- \frac{e^{-r\Delta}\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2(2(S_2-S_4)) + 2\frac{-\frac{\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}]. \end{aligned}$$

Rearranging, we see that

$$\frac{e^{-r\Delta}\gamma}{r}(1-e^{-r\Delta})(1-q)(1+c)^2(2S_4) + 2\frac{-\frac{\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2} = 2(1-e^{-r\Delta})\frac{\gamma}{r}[e^{-r\Delta}(1-q)(1+c)^2 S_4 - S_4],$$

where $e^{-r\Delta}(1-q)(1+c)^2 S_4 - S_4 = -1$. Pulling together S_2 terms and noting $(1+c)c - (1+c)^2 = -(1+c)$, we have

$$\begin{aligned} \beta_1 = &\left(\frac{1-e^{-r\Delta}(1-q)(1+c)}{1-e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}}\right) \times [e^{-r\Delta} S_2 \left((1-q)[\frac{c^2}{b} - 2\frac{\gamma}{r}(1-e^{-r\Delta})(1+c)]\right) \\ &- 2(1-e^{-r\Delta})\frac{\gamma}{r}]. \end{aligned}$$

Multiplying and dividing the last term by S_2, we arrive at

$$\beta_1 = \left(\frac{1 - e^{-r\Delta}(1-q)(1+c)}{1 - e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}} \right) \times [e^{-r\Delta} S_2 \left((1-q)\frac{c^2}{b} - 2\frac{\gamma}{r}(1 - e^{-r\Delta})e^{r\Delta} \right)],$$

and applying the definition of S_2,

$$\beta_1 = \frac{e^{-r\Delta} \left((1-q)\frac{c^2}{b} - 2\frac{\gamma}{r}(1 - e^{-r\Delta})e^{r\Delta} \right)}{1 - e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}}.$$

Finally, plug in the conjectured a, b, c, so that $\frac{c^2}{b} = (2\gamma/r)c$, and rearrange to find

$$\beta_1 = -2\frac{\gamma}{r} \frac{e^{-r\Delta} \left(-(1-q)c + (1 - e^{-r\Delta})e^{r\Delta} \right)}{1 - e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}} = -2\frac{\gamma}{r}.$$

Thus the conjectured equilibrium is an equilibrium (filling in the implied $\alpha_0^i, \alpha_2, \alpha_4$). Finally, note that

$$\frac{1-\hat{\omega}}{\Delta} = \frac{(n-1)(1-e^{-r\Delta}) + 2(1-q)e^{-r\Delta} - \sqrt{(n-1)^2(1-e^{-r\Delta})^2 + 4(1-q)e^{-r\Delta}}}{2(1-q)e^{-r\Delta}\Delta}.$$

Suppose that $q = \lambda\Delta$, so this becomes

$$\frac{1-\hat{\omega}}{\Delta} = \frac{(n-1)(1-e^{-r\Delta}) + 2(1-\lambda\Delta)e^{-r\Delta} - \sqrt{(n-1)^2(1-e^{-r\Delta})^2 + 4(1-\lambda\Delta)e^{-r\Delta}}}{2(1-\lambda\Delta)e^{-r\Delta}\Delta}.$$

Multiply the denominator and numerator by $e^{r\Delta}$ and take derivatives of the numerator and denominator:

$$\frac{(n-1)(e^{r\Delta}-1) + 2(1-\lambda\Delta) - \sqrt{(n-1)^2(1-e^{r\Delta})^2 + 4(1-\lambda\Delta)e^{r\Delta}}}{2(1-\lambda\Delta)\Delta}.$$

$$[2(1-2\lambda\Delta)]^{-1} \left((n-1)(re^{r\Delta}) - 2\lambda \right)$$

$$-.5\frac{\left((n-1)^2(1-e^{r\Delta})^2 + 4(1-\lambda\Delta)e^{r\Delta} \right)^{-.5} \left(-2re^{r\Delta}(n-1)^2(1-e^{r\Delta}) + 4r(1-\lambda\Delta)e^{r\Delta} - 4\lambda e^{r\Delta} \right)}{2(1-2\lambda\Delta)}.$$

Let $\Delta \to 0$:

$$\frac{1}{2}\left((n-1)r - 2\lambda\right) - .5\frac{(4)^{-.5}(4r - 4\lambda)}{2} = \frac{(n-2)r - \lambda}{2}.$$

We thus see that

$$\lim_{\Delta\to 0}\frac{-(1-\hat{\omega})}{\Delta}=\frac{-(n-2)r+\lambda}{2}$$

which is the instantaneous demand in the continuous time model. It is immediate that a, b converge to their corresponding limits, and since the strategies converge so too must the continuation values, for properly defined shocks.

G.2 Unobserved Z

Let the transfer $\hat{T}^i$ be defined exactly as in the continuous time model. As in the continuous time proof, in an equilibrium with truthtelling and affine δ, the transfers take the form

$$R_0 + R_1 Z_k + R_2 Z_k^2 + R_3 Z_k z_{ik} + R_4 z_{ik}.$$

The value function is thus linear-quadratic, so just as in the previous section, the equilibrium value function immediately after an auction or mechanism $V^+(z, Z)$ is linear quadratic in z, Z and thus can be rewritten

$$V^+(z, Z) = \upsilon_0 + \upsilon_1 z + \upsilon_2 \bar{Z} + \upsilon_3 z^2 + \upsilon_4 \bar{Z}^2 + \upsilon_5 z\bar{Z},$$

for constants $\upsilon_0 - \upsilon_5$. Then, following the steps of section D.4, maximizing

$$V^+(z + Y^i((\hat{z}^i, \hat{z}^{-i})), Z) + \hat{T}^i((\hat{z}^i, \hat{z}^{-i}); \phi)$$

is equivalent to maximizing

$$\begin{aligned}\mathcal{E}(\phi, Z, z^i, \hat{z}^i) \equiv{}& (\upsilon_1 + \upsilon_5\bar{Z})(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i) + \upsilon_3(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i)^2 \\ &+ 2\upsilon_3 z^i(\frac{Z - z^i}{n} - \frac{n-1}{n}\hat{z}^i) + \kappa_0(-n\delta(\phi) + Z - z^i + \hat{z}^i)^2 + \phi(\hat{z}^i - \delta(\phi)) + \frac{\phi^2}{4\kappa_0 n^2},\end{aligned}$$

Following the exact same steps as in the proof of proposition 5, we can show that $\mathcal{E}_\phi = z - \bar{Z}$ when evaluated at the equilibrium ϕ and $\hat{z}^i = z^i$, for the $\delta(\phi) = -\hat{a} - \hat{b}\phi$ consistent with equilibrium. Also, the equilibrium transfers must equal

$$(\upsilon_1 + (\upsilon_5 + 2\upsilon_3)\bar{Z})(z^i - \bar{Z}),$$

so it is straightforward to show the formulas for β_0, β_1 from the previous section apply again (for possibly different (a, b, c)).

Returning to the discrete time first order condition, the argument to be maximized when trader i submits an order x and report $\hat{z}^i$ is now

$$(1-q)\left(-x\frac{1}{-b(n-1)}\left((n-1)a + c(Z-z) + x\right) + V^+(z+x, Z)\right) + q\mathcal{E}(\phi, Z, z^i, \hat{z}^i)$$

Taking a derivative with respect to x, setting equal to 0, and using the result that $\mathcal{E}_\phi = z - \bar{Z}$

at the equilibrium $\phi, \hat{z}$,

$$(1-q)\left(-\phi+\frac{x}{b(n-1)}+(\beta_0+\beta_1\bar{Z})-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(z+x-\bar{Z})\right)-\frac{q}{b(n-1)}(z-\bar{Z})=0.$$

Plug in $x=a+b\phi+cz$, $\bar{Z}=\frac{-a-b\phi}{c}$, and $x=a+b\phi+cz$. The second order condition is met if and only if $b<0$. This also implies $x=c(z-\bar{Z})$, so

$$(z+x-\bar{Z})=(1+c)z+(1+c)\frac{a+b\phi}{c}.$$

The above can thus be rewritten

$$(1-q)\left(-\phi+\frac{a+b\phi+cz}{b(n-1)}+(\beta_0+\beta_1\frac{-a-b\phi}{c})-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}((1+c)z+(1+c)\frac{a+b\phi}{c})\right)$$
$$-\frac{q}{b(n-1)}(z+\frac{a+b\phi}{c})=0.$$

Gathering terms on $\phi, z, 1$:

$$0=(1-q)\left(-1+\frac{1}{n-1}-\frac{b\beta_1}{c}-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\frac{b}{c}\right)-\frac{q}{c(n-1)}$$
$$0=(1-q)\left(\frac{c}{b(n-1)}-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\right)-\frac{q}{b(n-1)}$$
$$0=(1-q)\left(\frac{a}{b(n-1)}+(\beta_0-\frac{a}{c}\beta_1)-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)(1+c)^2}(1+c)\frac{a}{c}\right)-\frac{qa}{bc(n-1)}.$$

We seek $a, b, c, \beta_1, \beta_0$ such that these three equations and the two equations defining β_0, β_1 all hold. Conjecture that for some $\tilde{\omega}\in(0,1)$, there is an equilibrium with $a=\frac{rv}{2\gamma}(1-\tilde{\omega})$, $b=-\frac{r}{2\gamma}(1-\tilde{\omega})$, and $c=-(1-\tilde{\omega})$. Starting with the coefficients on z, this means we need

$$0=(1-q)\left(\frac{2\gamma}{r(n-1)}-\frac{\frac{2\gamma}{r}(1-e^{-r\Delta})}{1-e^{-r\Delta}(1-q)\tilde{\omega}^2}\tilde{\omega}\right)+\frac{2\gamma q}{r(n-1)(1-\tilde{\omega})}.$$

Multiply through by $\frac{r}{2\gamma}$,

$$0=(1-q)\left(\frac{1}{(n-1)}-\frac{(1-e^{-r\Delta})\tilde{\omega}}{1-e^{-r\Delta}(1-q)\tilde{\omega}^2}\right)+\frac{q}{(n-1)(1-\tilde{\omega})}. \tag{98}$$

Suppose there exists a $\tilde{\omega}\in(0,1)$ such that this holds. Straightforward calculation shows that plugging in $\beta_0=v, \beta_1=\frac{-2\gamma}{r}$, the coefficients on $\phi, 1$ above all equal 0.

Following the steps in the last section, in any equilibrium, we have

$$\beta_1 = \frac{e^{-r\Delta}\left((1-q)\frac{c^2}{b} - 2\frac{\gamma}{r}(1-e^{-r\Delta})e^{r\Delta}\right)}{1-e^{-r\Delta}(1-q)(1+c) - qe^{-r\Delta}}.$$

Plugging in the conjectured a, b, c,

$$\beta_1 = \frac{e^{-r\Delta}\left(-\frac{2\gamma}{r}(1-q)(1-\tilde{\omega}) - 2\frac{\gamma}{r}(1-e^{-r\Delta})e^{r\Delta}\right)}{1-e^{-r\Delta}(1-q)\tilde{\omega} - qe^{-r\Delta}}.$$

For $\beta_1 = -\frac{2\gamma}{r}$ to be consistent, it must be that

$$1-e^{-r\Delta}(1-q)\tilde{\omega} - qe^{-r\Delta} = e^{-r\Delta}\left((1-q)(1-\tilde{\omega}) + (1-e^{-r\Delta})e^{r\Delta}\right),$$

but this holds for any $\tilde{\omega}$. Likewise, conjecturing that $\beta_0 = v$, at the conjectured a, b, c,

$$\begin{aligned} q\beta_0 + (1-q)[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)] &= qv + (1-q)[v(1-\tilde{\omega}) + v(1-e^{-r\Delta})\tilde{\omega}] \\ &= v\left(1-(1-q)\tilde{\omega}e^{-r\Delta}\right) \end{aligned}$$

and thus $\beta_0 = v$ is consistent with

$$\beta_0 = v(1-e^{-r\Delta}) + \frac{e^{-r\Delta}\left(q\beta_0 + (1-q)[\frac{ca}{b} + v(1-e^{-r\Delta})(1+c)]\right)}{1-e^{-r\Delta}(1-q)(1+c)}.$$

We have thus shown that, as long as $\tilde{\omega}$ satisfies (98), the conjectured a, b, c satisfy the first order condition and comprise a subgame perfect equilibrium. In unreported numerical exercises, we find for very small Δ there exists a root $\tilde{\omega}$ such that $-(1-\tilde{\omega})/\Delta$ is equal to the order flow coefficient c from proposition 5, up to machine precision error.