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On-line Appendix for "Coalition Formation in Legislative Bargaining"

Abstract

In this appendix we present the proofs and sections omitted in "Coalition Formation in Legislative Bargaining."

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1 Proof of Proposition 3

The share of output captured by i is $s_i = \frac{x^*_{f,i}}{V(C^*_f)}$. We have that:

$$\begin{aligned}\lim_{\Delta\to 0}\frac{x^*_{f,i}}{V(C^*_f)} &= \lim_{\Delta\to 0}\frac{u_i + \frac{1}{n(C^*_f)}\left[V(C^*_f) - \sum_{j\in C^*_f} u_j\right]}{V(C^*_f)} \\ &= \lim_{\Delta\to 0}\left[\frac{u_i}{V(C^*_f)} + \frac{1}{n(C^*_f)}\left(1 - \frac{\sum_{j\in C^*_f} u_j}{V(C^*_f)}\right)\right]\end{aligned}$$

It follows that as $\Delta \to 0$, $s_i \geq s_{i'} \iff \frac{u_i - u_{i'}}{V(C^*_f)} \geq 0$, that is if and only if $u_i \geq u_{i'}$. The parties' shares of output are non decreasing in the respective electoral sizes if and only if reservation utilities are non decreasing in the electoral sizes. ■

2 Proof of Case 2 in Proposition 4

We complete here the proof of Proposition 4 presenting the argument for case 2, in which $d < 0$, $e < 0$. We proceed in three steps: assuming first that 3 forms a government with 2; then that 3 forms a government with 1; and finally that 3 is unable to form a coalition.

Step 1. Assume that 3 forms a government with 2. We show that it can not be that 2 forms a government with 3. In this case we have:

$$\begin{aligned}S_2(\{2,3\}) &= \frac{1}{2}\left(a - x^3_2 - x^3_3\right) = 0 \\ S_2(\{1,2\}) &= \frac{1}{2}\left(a - d - x^3_2 - x^3_1\right) > S_2(\{2,3\})\end{aligned}$$

since $x^3_3 \geq 0$, $x^3_1 = 0$ and $d < 0$, a contradiction.

We now show that 2 must form a government with some other party. Assume not. Then, since, $x^3_1 = 0$ and $x^3_2 \leq a$, we have:

$$S_2(\{1,2\}) = \frac{1}{2}\left(a - d - x^3_1 - x^3_2\right) = \frac{1}{2}\left(a - d - x^3_2\right) \geq -d/2 > 0$$

Thus 2 can profitably form a government with 1, a contradiction.

We conclude that if 3 forms with 2, then 2 must form with 1. We now show that if 3 forms with 2 and 2 must form with 1, then 1 must be able to form a government. Assume not, then we have:

$$x^2_1 = \frac{a-d}{2} - \frac{x^3_2}{2},\ x^2_2 = \frac{a-d}{2} + \frac{x^3_2}{2},\ x^2_3 = 0$$

$$x_1^3 = 0,\ x_2^3 = \frac{a}{2} + \frac{x_2^2}{2},\ x_3^3 = \frac{a}{2} - \frac{x_2^2}{2}.$$

It follows that $x_2^2 = a - (2/3)d$ and $x_2^3 = a - d/3 > a$, impossible.

We now show that if 3 forms with 2 and 2 forms with 1, then 1 can not form a government with 2. In this case we have:

$$\begin{aligned}
x_1^1 &= \frac{a-d}{2} + \frac{x_1^2 - x_2^2}{2},\ x_2^1 = \frac{a-d}{2} - \frac{x_1^2 - x_2^2}{2},\ x_3^1 = 0 \\
x_1^2 &= \frac{a-d}{2} - \frac{x_2^3}{2},\ x_2^2 = \frac{a-d}{2} + \frac{x_2^3}{2},\ x_3^2 = 0 \\
x_1^3 &= 0,\ x_2^3 = \frac{a}{2} + \frac{x_2^1}{2},\ x_3^3 = \frac{a}{2} - \frac{x_2^1}{2}.
\end{aligned}$$

These equations imply that:

$$\begin{aligned}
x_1^1 &= -\frac{d}{3},\ x_2^1 = \frac{3a-2d}{3},\ x_3^1 = 0 \\
x_1^2 &= -\frac{d}{3},\ x_2^2 = a - \frac{2}{3}d,\ x_3^2 = 0 \\
x_1^3 &= 0,\ x_2^3 = a - \frac{d}{3},\ x_3^3 = \frac{d}{3}.
\end{aligned}$$

Note that $x_2^3 = a - \frac{d}{3} > a$ and $x_3^3 = \frac{d}{3} < 0$, but then 3 prefers to stay out the coalition $\{2,3\}$: a contradiction. We conclude that if 3 forms with 2, then 2 forms with 1 and 1 forms with 3.

Step 2. Assume now that 3 forms with 1. Assume first that 1 forms with 3. In this case we have:

$$\begin{aligned}
S_3(\{2,3\}) &= \frac{1}{2}\left(a - x_3^1 - x_2^1\right) = \frac{1}{2}\left(a - x_3^1\right) \\
&> \frac{1}{2}\left(a + e - x_1^1 - x_3^1\right) = S_3(\{1,3\})
\end{aligned}$$

a contradiction. We now show that 1 must form a government with 2. Assume first that neither 1 nor 2 forms a government, then $x_2^2 = 0$. Moreover we must have $x_1^2 \leq a + e$. We have: $S_1(\{1,2\}) = \frac{1}{2}\left(a - d - x_1^2\right) \geq \frac{1}{2}\left(a - d - (a+e)\right) = -(d+e)/2 > 0$, a contradiction.

Assume now that 1 is unable to form and 2 forms with 3. We must have:

$$\begin{aligned}
x_1^2 &= 0,\ x_2^2 = \frac{a}{2} - \frac{x_3^3}{2},\ x_3^2 = \frac{a}{2} + \frac{x_3^3}{2} \\
x_1^3 &= \frac{a+e}{2} - \frac{x_3^2}{2},\ x_2^3 = 0,\ x_3^3 = \frac{a+e}{2} + \frac{x_3^2}{2}.
\end{aligned}$$

This gives us $x_3^3 = a + (2/3)e > a + e$, a contradiction.

Finally, assume that 1 is unable to form and 2 forms with 1. In this case we have:

$$\begin{aligned} x_1^2 &= \frac{a-d}{2} + \frac{x_1^3}{2},\, x_2^2 = \frac{a-d}{2} - \frac{x_1^3}{2},\, x_3^2 = 0 \\ x_1^3 &= \frac{a+e}{2} + \frac{x_1^2}{2},\, x_2^3 = 0,\, x_3^3 = \frac{a+e}{2} - \frac{x_1^2}{2}. \end{aligned}$$

This gives us $x_1^2 = \frac{3a-2d+e}{3}$, $x_1^3 = \frac{3a+2e-d}{3}$ and $x_3^3 = \frac{d+e}{3} < 0$.

We conclude that if 3 forms with 1 then 1 forms with 2. Assume now that 2 is unable to form. We have:

$$\begin{aligned} x_1^1 &= \frac{a-d}{2} + \frac{x_1^3}{2},\, x_2^1 = \frac{a-d}{2} - \frac{x_1^3}{2},\, x_3^1 = 0 \\ x_1^3 &= \frac{a+e}{2} + \frac{x_1^1}{2},\, x_2^3 = 0,\, x_3^3 = \frac{a+e}{2} - \frac{x_1^1}{2}. \end{aligned}$$

So $x_1^1 = a + \frac{e-2d}{3}$ and $x_3^3 = \frac{a+e}{2} - \frac{a}{2} - \frac{e-2d}{6} = \frac{2e+2d}{6} < 0$, a contradiction.

Finally assume that 3 forms with 1, 1 forms with 2 and 2 forms with 1. We have:

$$\begin{aligned} x_1^1 &= \frac{a-d}{2} + \frac{x_1^2 - x_2^2}{2},\, x_2^1 = \frac{a-d}{2} - \frac{x_1^2 - x_2^2}{2},\, x_3^1 = 0 \\ x_1^2 &= \frac{a-d}{2} + \frac{x_1^3}{2},\, x_2^2 = \frac{a-d}{2} - \frac{x_1^3}{2},\, x_3^2 = 0. \\ x_1^3 &= \frac{a+e}{2} + \frac{x_1^1}{2},\, x_2^3 = 0,\, x_3^3 = \frac{a+e}{2} - \frac{x_1^1}{2} \end{aligned}$$

so $x_1^1 = a + \frac{e-2d}{3}$ and $x_3^3 = \frac{1}{3}(e+d) < 0$, a contradiction.

Step 3. We now show that if 3 is unable to form a government, then no other party forms a government with 3, thus a coalition $\{1,2\}$ forms. We have 2 cases to rule out:

Case 1. Assume that 3 is unable to form a government and 1 forms $\{1,3\}$. If 2 is unable to form a government too, then we have $S_1(1,2) = (a-d-x_1^1)/2$ which is larger than $S_1(1,3) = (a+e-x_1^1-x_3^1)/2$.

If 2 forms a government with 3, we have:

$$\begin{aligned} x_1^1 &= \frac{a+e-x_3^2}{2}, x_2^1 = 0, x_3^1 = \frac{a+e+x_3^2}{2} \\ x_1^2 &= 0, x_2^2 = \frac{a-x_3^1}{2}, x_3^2 = \frac{a+x_3^1}{2} \\ x_1^3 &= x_1^1, x_2^3 = 0, x_3^3 = x_3^1 \end{aligned}$$

which after solving gives: $x_3^2 = a + \frac{e}{3}$, $x_1^1 = \frac{e}{3} < 0$, a contradiction.

If 2 forms a government with 1, we have that:

$$
\begin{aligned}
x_1^1 &= \frac{a+e}{2} + \frac{x_1^2}{2},\ x_2^1 = 0,\ x_3^1 = \frac{a+e}{2} - \frac{x_1^2}{2} \\
x_1^2 &= \frac{a-d}{2} + \frac{x_1^1}{2},\ x_2^2 = \frac{a-d}{2} - \frac{x_1^1}{2},\ x_3^2 = 0.
\end{aligned}
$$

Which gives us:

$$
\begin{aligned}
x_1^1 &= \frac{3a+2e-d}{3},\ x_2^1 = 0,\ x_3^1 = \frac{e+d}{3} \\
x_1^2 &= \frac{3a+e-2d}{3},\ x_2^2 = -\frac{e+d}{3},\ x_3^2 = 0.
\end{aligned}
$$

but then $x_3^1 = (e+d)/3 < 0$, a contradiction.

Case 2. Assume that 3 is unable to form a government and 2 forms $\{2,3\}$. If 1 is unable to form a government too, then we have $x_1^2 = 0$ and $x_2^2 \leq a$. It follows that $S_1(1,2) = (a-d-x_1^2-x_2^2)/2 \geq -d/2 > 0$, thus 1 forms a coalition with 2, a contradiction. If 1 forms a government with 3, we have:

$$
\begin{aligned}
x_1^1 &= \frac{a+e}{2} - \frac{x_3^2}{2},\ x_2^1 = 0,\ x_3^1 = \frac{a+e}{2} + \frac{x_3^2}{2} \\
x_1^2 &= 0,\ x_2^2 = \frac{a}{2} - \frac{x_3^1}{2},\ x_3^2 = \frac{a}{2} + \frac{x_3^1}{2}.
\end{aligned}
$$

After solving the system above, we have $x_1^1 = \frac{e}{3} < 0$, a contradiction.

Finally, if 1 forms a government with 2, we have:

$$
\begin{aligned}
x_1^1 &= \frac{a-d}{2} - \frac{x_2^2}{2},\ x_2^1 = \frac{a-d}{2} + \frac{x_2^2}{2},\ x_3^1 = 0 \\
x_1^2 &= 0,\ x_2^2 = \frac{a}{2} + \frac{x_2^1}{2},\ x_3^2 = \frac{a}{2} - \frac{x_2^1}{2}.
\end{aligned}
$$

Solving the system, we obtain:

$$
\begin{aligned}
x_1^1 &= -\frac{d}{3},\ x_2^1 = \frac{3a-2d}{3},\ x_3^1 = 0 \\
x_1^2 &= 0,\ x_2^2 = \frac{3a-d}{3},\ x_3^2 = \frac{d}{3}.
\end{aligned}
$$

But then $x_3^2 = d/3 < 0$, a contradiction. ■

From the Steps presented above, we conclude that either 3 is able to form a coalition, in which case the equilibrium is clockwise or counter-clockwise; or 3 is unable to form a coalition, in which case the only coalition that can form is $\{1,2\}$, the efficient coalition with $d, e < 0$. ■

3 Proof of Proposition 5

We complete the proof of Proposition 5 by proving Lemmata A.5.2 and A.5.4.

Lemma A.5.2. Assume d and e are nonpositive. *If* $d > -(a+e)$*, then 3 forms a coalition with* 1 *or 2, when given proposal power. If* $d \leq -(a+e)$ *then there is an efficient equilibrium in which 3 is excluded from any coalition; and* 1 *and* 2 *receive a share* x_1, x_2 *for any* x_1, x_2 *such that* $x_1 \geq a+e, x_2 \geq a, x_1+x_2 \leq a-d$.

Proof. We have two cases to consider:

Case 1. We first consider the case $d > -(a+e)$. Assume that when given proposal power, 3 fails to form a government. Then we must have:

$$\begin{aligned} a+e-x_1^1-x_3^1 &\leq 0 \quad (1) \\ a-x_2^1-x_3^1 &\leq 0 \end{aligned}$$

else 3 would be able and find it profitable to form a government with either 1 or 2. As shown in Section 9.5.2 (Step 3, Case 2), if 3 is unable to form a coalition, then no other party chooses to forms a government with 3, so $x_3^1 = 0, x_3^2 = 0$. It follows that $x_1^1 \geq a+e$, and $x_2^1 \geq a$. Note that $x_1^1 + x_2^1 \leq a - d$. We conclude that an equilibrium in which 3 is never in a coalition can occur only if $d \leq -(a+e)$.

Case 2. Consider now the case $d \leq -(a+e)$. Let

$$X^* = \{x_1, x_2 \,| x_1 \geq a+e, x_2 \geq a, x_1+x_2 \leq a-d\}$$

It is easy to verify that here is an equilibrium in which 3 is unable to form a coalition and never included in any coalition by others; 1 proposes to 2 and 2 proposes to 1; $x_i^1 = x_i^2 = x_i$ for $i = 1,2$ and $x_1, x_2 \in X^*$; and $x_3^j = 0$ $j = 1,2$. ■

We now turn to Lemma A.5.4.

Lemma A.5.4. *An equilibrium in which* 1 *forms a coalition with 2, 2 with 3 and 3 with* 1 *(counterclockwise equilibrium) exists if and only if:* $d \leq \frac{3}{7}a - \frac{5}{7}e$ if $d, e > 0$; and $d \geq -\frac{3}{5}a - \frac{1}{5}e$ and $d \leq 3a + 7e$ if $d, e < 0$.

Proof. As in the previous lemma, we proceed in three steps.

Step 1. We first construct the value functions associated to a counterclockwise equilibrium. If a counterclockwise equilibrium exists, we must have:

$$
\begin{aligned}
x_1^1 &= \frac{a-d}{2} - \frac{x_2^2}{2},\ x_2^1 = \frac{a-d}{2} + \frac{x_2^2}{2},\ x_3^1 = 0 \\
x_1^2 &= 0,\ x_2^2 = \frac{a}{2} - \frac{x_3^3}{2},\ x_3^2 = \frac{a}{2} + \frac{x_3^3}{2} \\
x_1^3 &= \frac{a+e}{2} + \frac{x_1^1}{2},\ x_2^3 = 0,\ x_3^3 = \frac{a+e}{2} - \frac{x_1^1}{2},
\end{aligned}
$$

Which implies:

$$
\begin{aligned}
x_1^1 &= \frac{3a-4d+e}{9},\ x_2^1 = \frac{6a-5d-e}{9},\ x_3^1 = 0 \\
x_1^2 &= 0,\ x_2^2 = \frac{3a-d-2e}{9},\ x_3^2 = \frac{6a+d+2e}{9} \\
x_1^3 &= \frac{6a-2d+5e}{9},\ x_2^3 = 0,\ x_3^3 = \frac{3a+2d+4e}{9},
\end{aligned}
$$

Step 2. To check the conditions under which this equilibrium exists if $d, e > 0$, consider first the case in which 1 is the formateur.

$$
\begin{aligned}
S_1(\{1,2\}) &= \frac{1}{2}\left(a-d-x_1^2-x_2^2\right) = \frac{6a+2e-8d}{18} > 0 \\
S_1(\{1,3\}) &= \frac{1}{2}\left(a+e-x_2^2-x_3^2\right) = \frac{3a+7e-d}{18}
\end{aligned}
$$

It follows that $S_1(\{1,2\}) \geq S_1(\{1,3\})$ if and only if $d \leq \frac{3}{7}a - \frac{5}{7}e$. Moreover, $S_1(\{1,2\}) \geq 0$ if $d \leq \frac{3}{4}a + \frac{e}{4}$which is implied by $d \leq \frac{3}{7}a - \frac{5}{7}e$ for $d > 0$.

Consider now Formateur 2. We have:

$$
\begin{aligned}
S_2(\{1,2\}) &= \frac{1}{2}\left(a-d-x_1^3\right) = \frac{3a-7d-5e}{18} \geq 0 \\
S_2(\{2,3\}) &= \frac{1}{2}\left(a-x_3^3\right) = \frac{6a-2d-4e}{18}
\end{aligned}
$$

so $S_2(\{2,3\}) > S_2(\{1,2\})$ is always true.

Finally, consider now Formateur 3's decision.

$$
\begin{aligned}
S_3(\{1,3\}) &= \frac{1}{2}\left(a+e-x_1^1-x_3^1\right) = \frac{6a+8e+4d}{18} > 0 \\
S_3(\{2,3\}) &= \frac{1}{2}\left(a-x_2^1-x_3^1\right) = \frac{3a+5d+e}{18}
\end{aligned}
$$

so $S_3(\{1,3\}) > S_3(\{2,3\})$ is true if $d < a + 7e$, which holds whenever $d \leq \frac{3}{7}a - \frac{5}{7}e$. We conclude that a clockwise equilibrium exists if and only if $d \leq \frac{3}{7}a - \frac{5}{7}e$.

Step 3. To check the conditions under which this equilibrium exists if $d, e < 0$, consider first the case in which 2 is the formateur. We have:

$$\begin{aligned} S_2(\{1,2\}) &= \frac{1}{2}\left(a - d - x_1^3\right) = \frac{3a - 7d - 5e}{18} > 0 \\ S_2(\{2,3\}) &= \frac{1}{2}\left(a - x_3^3\right) = \frac{6a - 2d - 4e}{18} \end{aligned}$$

so $S_2(\{2,3\}) \geq S_2(\{1,2\})$ if $d \geq -\frac{3}{5}a - \frac{1}{5}e$.

Consider now Formateur 1. We have:

$$\begin{aligned} S_1(\{1,2\}) &= \frac{1}{2}\left(a - d - x_2^2\right) = \frac{6a - 8d + 2e}{18} > 0 \\ S_1(\{1,3\}) &= \frac{1}{2}\left(a + e - x_3^2\right) = \frac{3a + 7e - d}{18} \end{aligned}$$

It follows that $S_1(\{1,2\}) > S_1(\{1,3\})$ if and only if $3a - 7d - 5e \geq 0$.

Finally, consider now Formateur 3's decision.

$$\begin{aligned} S_3(\{1,3\}) &= \frac{1}{2}\left(a + e - x_1^1\right) = \frac{6a + 4d + 8e}{18} \\ S_3(\{2,3\}) &= \frac{1}{2}\left(a - x_2^1\right) = \frac{3a + 5d + e}{18} \end{aligned}$$

It follows that $S_3(\{1,3\}) \geq S_3(\{2,3\})$ if $3a + 7e - d \geq 0$. Note that $3a - 7d + 5e > 3a + 7e - d$ for $d < 0$, thus the condition on formateur 3 implies the condition on formateur 1. We conclude that a counter-clockwise equilibrium exists if and only if $d \geq -\frac{3}{5}a - \frac{1}{5}e$ and $d \leq 3a + 7e$ ■

4 Proof of Proposition 6

Assume first that $d, e \geq 0$, so $\theta \geq 0$. From Proposition 5 we know that we necessarily have a positive bonus over the other coalition members if the clock-wise equilibrium is unique. This happens if and only if $d < a - e$, $d > \frac{3}{7}a - \frac{5}{7}e$. The first condition is satisfied if and only if $|\theta| < \theta^F_{**}$, the second if $|\theta| > \theta^F_*$. Similarly for $d, e \leq 0$ it can be shown that the clock wise equilibrium is the unique equilibrium if and only if $d > -a - e$ and $d < -\frac{3}{5}a - \frac{1}{5}e$ and $d > 3a + 7e$. This again is satisfied if and only if $|\theta| < \theta^F_{**}$, the second if $|\theta| > \theta^F_*$. ■

5 Proof of Proposition 7

From conditions (13) in the paper, a "grand coalition" equilibrium in which the grand coalition forms independently of the order of formateurs exists if and only if $g \geq \max_{z \leq e+g} \{a - d - z\}$, i.e. the grey area in Figure 4 is non empty. It is immediate to verify that this is true if and only if $g \geq (a - d - e)/2$ as stated. ■

6 Proof of Proposition 8

The calculations of the thresholds θ_*^P and θ_{**}^P are analogous as the calculations of θ_*^F and θ_{**}^F in Proposition 7. It follows from the discussion in the text that if $\theta \geq \theta_{**}^P$ the head of state is irrelevant and if $\theta < \theta_*^P$ the head of state has incomplete control, as defined in Section 6. For $\theta \in \left(\theta_*^P, \theta_{**}^P\right]$ there are two cases. For $\theta \in \left(\theta_*^P, \theta_{***}^P\right]$, Proposition 5 shows that there is a unique clockwise equilibrium in which an efficient outcome can be achieved by selecting 1 as first formateur (if $\theta > 0$) or 2 (if $\theta < 0$). If instead $\theta \in \left(\theta_{***}^P, \theta_{**}^P\right]$, it follows from Propositions 4 and 5 that there is no pure strategy equilibrium. In Section 7.2 and the online appendix we show that the head of state can achieve an efficient equilibrium selecting order $1 \to 3 \to 2$ starting with 3 with $\theta > 0$; and and order $1 \to 2 \to 3$ starting with 2 with $\theta < 0$. ■

7 Proof of Proposition 9

We first characterize the set in which the head of state can achieve an efficient outcome in all equilibria, by committing to some order of formateurs. By Proposition 8, the efficient outcome is achieved in any equilibrium and the head of state is not irrelevant if and only if the clockwise equilibrium exists, but the counterclockwise does not exist. There are two cases to consider.

Case 1. $d, e > 0$. The head of state has two options: order $1 \to 2 \to 3$ and $1 \to 3 \to 2$.[1] As before, in the first case, the head of state is not irrelevant and the efficient outcome can be implemented in all equilibria if and only if the clockwise equilibrium exists and the count-clockwise does not. A clockwise equilibrium with order $1 \to 2 \to 3$ and $d, e > 0$ exists only if $d \leq \min\left(\frac{3}{4}a + \frac{e}{4}, 3a - 2e\right)$

[1] Recall from Section 6 that with the notation $i \to j \to k$ we mean that j is formateur after i, k after j, and i either is first or comes after k.

and a counterclockwise does not exist if $d \geq \frac{3}{7}a - \frac{5}{7}e$. In the second case, first note that a clockwise equilibrium exists with $1 \to 3 \to 2$ if and only if a counter-clockwise equilibrium exists in a game with order $1 \to 2 \to 3$ and $V(\{1,2\}) = a + e$ and $V(\{1,3\}) = a - d$ (to see this, just switch the labels of 2 and 3): in this case, however, by Proposition 5, a clockwise equilibrium also exists. It follows that the efficient outcome can be implemented as the unique outcome and the head of state is not irrelevant only when the counter-clockwise equilibrium exists and the clockwise equilibrium does not exist in the game with order $1 \to 3 \to 2$. A counter-clockwise equilibrium in a game with order $1 \to 3 \to 2$ and $V(\{1,2\}) = a - d$ and $V(\{1,3\}) = a + e$ with $d, e > 0$ exists if and only if a clockwise equilibrium exists in a game with order $1 \to 2 \to 3$ and $V(\{1,2\}) = a + e$ and $V(\{1,3\}) = a - d$ with $d, e > 0$ (again, to see this, just switch the labels of 2 and 3). From Proposition 5 we have that in this game a clockwise equilibrium exists if:

$$\begin{aligned} a - V(\{1,2\}) &\geq -\frac{3}{2}a - 2\left[V(\{1,2\}) - e\right] \\ &\Leftrightarrow d \leq \frac{3}{4}a - \frac{e}{2} \end{aligned}$$

Similarly a counter-clockwise equilibrium in a game with order $1 \to 2 \to 3$ and $V(\{1,2\}) = a + e$ and $V(\{1,3\}) = a - d$ with $d, e > 0$ does not exists if and only if:

$$\begin{aligned} a - V(\{1,2\}) &\leq -\frac{3}{5}a - \frac{V(\{1,3\} - a)}{5} \text{ or} \\ a - V(\{1,2\}) &\geq 3a + 7\left[V(\{1,3\} - a)\right] \end{aligned}$$

That is: $d \geq 3a - 5e$ or $d \leq \frac{3}{7}a + \frac{e}{7}$. It is easy to verify that the set in which the head of state is not irrelevant and can achieve the efficient outcome as the unique equilibrium with order $1 \to 3 \to 2$ is a subset of the set with order $1 \to 2 \to 3$. When $d, e > 0$, the required set is therefore defined by $d \leq \min\left(\frac{3}{4}a + \frac{e}{4}, 3a - 2e\right)$ and $d \geq \frac{3}{7}a - \frac{5}{7}e$.

Case 2. $d, e < 0$. Again, the head of state has two options: order $1 \to 2 \to 3$ and $1 \to 3 \to 2$. In the first case, the head of state is not irrelevant and the efficient outcome can be implemented in all equilibria if only if the clockwise equilibrium exists and the counterclockwise does not exist. A clockwise equilibrium with order $1 \to 2 \to 3$ and $d, e < 0$ exists only if $d \geq -\frac{3}{2}a - 2e$ and a counterclockwise does not exist if $d \leq \frac{3}{5}a - \frac{1}{5}e$ and $d \geq 3a + 7e$. In the second case (order $1 \to 3 \to 2$), again note that a clockwise equilibrium exists if and only if a counter-clockwise equilibrium exists in a game with order $1 \to 2 \to 3$ and $V(\{1,2\}) = a + e$ and $V(\{1,3\}) = a - d$: in this case, however, by Proposition 5, a clockwise equilibrium also exists. We therefore need that the counter-clockwise equilibrium exists and the clockwise equilibrium does not in the game with order $1 \to 3 \to 2$. Now note that a counter-clockwise equilibrium in a game with order $1 \to 3 \to 2$ and $V(\{1,2\}) = a - d$ and $V(\{1,3\}) = a + e$ with $d, e < 0$ exists if and only if a clockwise equilibrium exists in a game with order $1 \to 2 \to 3$ and $V(\{1,2\}) = a + e$ and $V(\{1,3\}) = a - d$. From Proposition 5 we have that in this game a clockwise equilibrium exists if:

$$\begin{aligned} a - V(\{1,2\}) &\leq \min\left(\frac{3}{4}a + \frac{V(\{1,3\} - a}{4}, 3a - 2\left[V(\{1,3\} - a\right]\right) \\ &\Leftrightarrow \quad -e \leq \min\left(\frac{3}{4}a - \frac{d}{4}, 3a + 2d\right) \\ &\Leftrightarrow \quad d \leq 3a + 4e \text{ and } d \geq -\frac{3}{2}a - \frac{e}{2} \end{aligned}$$

Similarly, a counter-clockwise equilibrium does not exists if and only if:

$$\begin{aligned} a - V(\{1,2\}) &\geq \frac{3}{7}a - \frac{5}{7}\left[V(\{1,3\} - a)\right] \\ &\Leftrightarrow \quad -e \geq \frac{3}{7}a + \frac{5}{7}d \\ &\Leftrightarrow \quad d \leq -\frac{3}{5}a - \frac{7}{5}e \end{aligned}$$

It is easy to verify that the set with order $1 \to 2 \to 3$ is a subset of the set with order $1 \to 3 \to 2$. When $d, e < 0$, the set in which the head of state is not irrelevant and can achieve an efficient outcome in all equilibria committing to an order of formateurs is therefore defined by $d \leq \min\left\{3a + 4e, -\frac{3}{5}a - \frac{7}{5}e\right\}$ and $d \geq -\frac{3}{2}a - \frac{e}{2}$.

Regarding the definition of the set of parameters in which the head of state's selection is time consistent, the argument presented in Section 6 makes clear that when $d, e > 0$ the order $1 \to 2 \to 3$ is not time consistent, but $1 \to 3 \to 2$ is. When $d, e > 0$, therefore, the set in which the head of state is not irrelevant and can achieve an efficient outcome in all equilibria by selecting a time consistent order of formateurs is therefore defined by $d \leq \frac{3}{4}a - \frac{e}{2}$ and $d \geq 3a - 5e$ or $d \leq \frac{3}{7}a + \frac{e}{7}$. By a similar argument, when $d, e < 0$ the order $1 \to 3 \to 2$ is not time consistent, but $1 \to 2 \to 3$ is. When $d, e < 0$, therefore, the set in which the head of state is not irrelevant and can achieve an efficient outcome in all equilibria by selecting a time consistent order of formateurs is defined by $d \geq \max\left\{-\frac{3}{2}a - 2e, 3a + 7e\right\}$ and $d \leq -\frac{3}{5}a - \frac{1}{5}e$. The result presented in the main text is obtained immediately by rewriting these conditions in the notation used in Section 6, that is, when $\theta > 0$, $\theta \leq 3/\left(4\vartheta + 2\epsilon\right)$, and $\theta \geq 3/\left(\vartheta + 5\epsilon\right)$ or $\theta \leq 3/\left(7\vartheta - \epsilon\right)$; when $\theta < 0$, $\theta \leq 3 \cdot \max\left\{1/(\vartheta - 7\epsilon), -1/(2\vartheta + 4\epsilon)\right\}$ and $\theta \geq -3/\left(5\vartheta + \epsilon\right)$. ■

8 The mixed strategy equilibrium of Section 7.2

We construct a mixed strategy equilibrium with which the head of state can achieve an efficient outcome in the two regions in which a pure strategy equilibrium does not exist. We proved in two steps.

Step 1: Assume first that $d, e < 0$. With $d, e < 0$ a pure strategy equilibrium does not exist if $d < -\frac{3}{2}a - 2e$ and $d > -(a+e)$. For this region we construct an equilibrium in which 1 chooses 2 with probability α, and 3 with probability $1-\alpha$; 2 chooses 1 with probability 1; and 3 chooses 2 with probability 1 (see lower right panel in Figure 2). Let us assume an order of formateurs in which 2 follows 1, 3 follows 2, and 1 follows 3 (in short $1 \to 2 \to 3$). We must have:

$$x_3^1 = \frac{a+e}{2} - \frac{x_1^2}{2},\ x_2^1 = \frac{a-d}{2} + \frac{x_2^2 - x_1^2}{2},$$

The indifference condition for 1 is:

$$\frac{a+e}{2} + \frac{x_1^2}{2} = \frac{a-d}{2} - \frac{x_2^2 - x_1^2}{2}$$

Implying $x_2^2 = -(e+d)$, $x_1^2 = a - d - x_2^2 = a + e$, $x_3^2 = 0$.

We also must have $x_3^1 = 0$, $x_1^1 = \frac{a+e}{2} + \frac{x_1^2}{2} = a + e$ and $x_2^1 = a - d - x_1^1 = -(e+d)$. Moreover, $x_1^3 = 0$,

$$\begin{aligned} x_2^3 &= \frac{a}{2} + \frac{\alpha x_2^1 - (1-\alpha)x_3^1}{2} = \frac{a}{2} - \frac{\alpha}{2}x_2^1 \\ &= \frac{a}{2} - \frac{\alpha}{2}(e+d) \end{aligned}$$

And we have:

$$x_2^2 = \frac{a-d}{2} + \frac{x_2^3 - x_1^3}{2} = \frac{a-d}{2} + \frac{x_2^3}{2}$$

Thus:

$$\frac{a-d}{2} + \frac{x_2^3}{2} = -(d+e)$$

Implying $x_2^3 = -(d + 2e + a)$. Substituting in the formula above we have:

$$\alpha = \frac{3a + 2d + 4e}{e + d}$$

Note that since $d + e < 0$, $\alpha \geq 0$ for $d \leq -\frac{3}{2}a - 2e$. Moreover $\alpha \leq 1$ for $3a + 2d + 4e \leq e + d$, that is if $d \geq -3(a+e)$. It can be verified tat these conditions are always satisfied in the region of interest.

Obviously the strategy is optimal for formateur 1 by construction. For formateur 2, it is optimal if $S_2(1,2) \geq S_2(2,3)$. We have:

$$\begin{aligned} S_2(1,2) &= \frac{1}{2}(a - d - x_1^3 - x_2^3) = 2(a+e) \geq 0 \\ S_2(2,3) &= \frac{1}{2}(a - x_2^3 - x_3^3) = 0 \end{aligned}$$

The condition is therefore verified.

For formateur 3 we need $S_3(2,3) \geq S_3(1,3)$. We have:

$$\begin{aligned} S_3(1,3) &= \frac{1}{2}(a + e - x_1^1 - Ex_3^1) = 0 \\ S_3(2,3) &= \frac{1}{2}(a - Ex_2^1 - Ex_3^1) = a - \alpha(e+d) \\ &= \frac{1}{2}\left[a + \frac{3a+2d+4e}{e+d}(e+d)\right] = 2a + d + 2e \end{aligned}$$

Thus the condition is verified if $d \geq -2(a+e)$. Note that in the relevant region we need to have: $d \geq -\frac{3}{2}a - 2e > -2(a+e)$, thus this condition is always satisfied. Given this equilibrium, an efficient outcome in which {1,2} is selected with probability one is obtained by the head of state by choosing an order $2, 1, 3$.

Step 2: Assume now that $d, e > 0$. With $d, e > 0$ a pure strategy equilibrium does not exist if $d < a - e$ and $d > \frac{3}{4}a + e/4$. Now, note that if there is an equilibrium with strategies as described in Step 1 and order $1 \to 2 \to 3$, then there is an equilibrium in which 1 selects 3 with probability α and selects 2 with probability $1 - \alpha$; party 3 selects 1 and party 2 selects 3 with probability one for the game in which the order is $1 \to 3 \to 2$ and $V(\{1,2\}) = a + e$ and $V(\{1,3\}) = a - d$ (to see this, just switch the labels of 2 and 3). This equilibrium exists if:

$$\begin{aligned} a - V(\{1,2\}) &\geq -2a - 2\left(V(\{1,3\} - a)\right)) \\ &\Leftrightarrow d \leq a - \frac{e}{2} \end{aligned}$$

and if:

$$\begin{aligned} a - V(\{1,2\}) &\leq -\frac{3}{2}a - 2\left(V(\{1,3\}) - a\right) \\ &\Leftrightarrow d \geq \frac{3}{4}a - \frac{e}{2} \end{aligned}$$

Which is define a superset of the area defined by $d < a - e$ and $d > \frac{3}{4}a + e/4$. The head of state can therefore implement an efficient outcome in which {1,3} is selected if the order is $1 \to 3 \to 2$ and party 3 is the first formateur. ■